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On an Impulsive Control of Additive Processes

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Let $(\Omega, \mathscr{B}, \mathscr{B}_t, P_x, X_t, \theta_t)$ be a temporally homogeneous Lévy Process on R^n (spatially homogeneous Hunt process on R^n). We assume that the process is of function space type and has a reference measure (cf. Appendix and [2]). Take a sequence of $\mathscr{B}_t$-stopping times $\{\tau_k\}_k$ such that

$$\tau_k \leqq \tau_{k+1} \quad \text{for each } k \tag{0.1}$$

where $<$ holds unless $\tau_k = \infty$, and

$$\lim_{k\to\infty} \tau_k = \infty \quad \text{a.s.}, \tag{0.2}$$

and a sequence of $[0,\infty)^n$-valued random variables $\{\xi_k\}_k$ such that each ξ_k is $\mathscr{B}_{\tau_k}$-measurable. We call the sequence $v = \{\tau_1, \xi_1; \tau_2, \xi_2; \ldots\}$ of such pairs $\{\tau_k, \xi_k\}$ an *impulsive control* and denote by V the class of all impulsive controls. Put

$$Y_t^v(\omega) = X_t(\omega) + \sum_{\tau_i(\omega) \leqq t} \xi_i(\omega) \tag{0.3}$$

and

$$u(x) = \inf_{v\in V} E_x\left[\int_0^\infty e^{-\alpha t} f(Y_t^v(\omega))\, dt + \sum_{i=1}^\infty e^{-\alpha\tau_i} k(\xi_i)\right] \tag{0.4}$$

where f and k are given nonnegative functions on R^n and $[0,\infty)^n$ respectively. We are concerned with finding an optimal control $v \in V$ in the above variational problem (0.4). A similar problem has been considered by A. Bensoussan, J.L. Lions [1]. They have shown the relation between some quasi-variational inequality and the impulsive control problem. More precisely, under the assumption that the solution of the quasi-variational inequality satisfies some regularity conditions, they proved that this solution turns out to be the minimal cost function corresponding to (0.4) and moreover that there exists an optimal control. But they did not discuss under what conditions their regularity assumptions are satisfied.

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In this paper, we first study the (uniform) continuity of the function $u(x)$ given by (0.4) and then deduce the following properties of $u(x)$:

$$u(x) \leqq E_x\left[\int_0^\tau e^{-\alpha t} f(X_t)\,dt + e^{-\alpha\tau} u(X_\tau)\right] \tag{0.5}$$

for each stopping time τ, and

$$u(x) \leqq \inf_{\xi \geqq 0} \{k(\xi) + u(x+\xi)\}. \tag{0.6}$$

Here $\xi \geqq 0$ means that each component of ξ is nonnegative. Inequalities (0.5) and (0.6) express some strategical meanings. By making use of these inequalities, it becomes clear that the above impulsive control problem is closely related to the following optimal stopping problem:

$$w(x) = \inf_\tau E_x\left[\int_0^\tau e^{-\alpha t} f(X_t)\,dt + e^{-\alpha\tau} M_u(X_\tau)\right] \tag{0.7}$$

where

$$M_u(x) = \inf_{\xi \geqq 0} \{k(\xi) + u(x+\xi)\} \tag{0.8}$$

and τ ranges over all $\mathscr{B}_t$-stopping times (Theorem 1). Moreover we note that the minimal cost function $u(x)$ is the largest of functions satisfying inequalities (0.5), (0.6) and some regularity conditions (cf. Remark at the end of § 3). Finally we show the existence of an optimal control as the consequence of the above observations (Theorem 2).

Now we state our result. Under the assumptions

(A.1) α is a positive constant,

(A.2) $f(x)$ is a uniformly continuous bounded nonnegative function, and

(A.3) $k(\xi)$ is nonnegative Borel function defined on $[0,\infty)^n$ such that

$$k(\xi_1) + k(\xi_2) - k(\xi_1 + \xi_2) \geqq c > 0 \quad \text{for each } \xi_1, \xi_2 \in [0,\infty)^n$$

where c is a positive constant, we have the following:

Theorem 1. $u(x) = w(x)$.

Theorem 2. *If there exists a Borel function $\xi(x)$ such that*

$$k(\xi(x)) + u(x+\xi(x)) = \inf_{\xi \geqq 0} \{k(\xi) + u(x+\xi)\}, \tag{0.9}$$

then the control $\hat{v} = \{\hat{\tau}_1, \hat{\xi}_1; \hat{\tau}_2, \hat{\xi}_2; \ldots\}$ defined below is optimal:

$$\begin{cases} \hat{\tau}_1(\omega) = \inf\{t \geqq 0; X_t \in D\} \\ \hat{\xi}_1(\omega) = \xi(X_{\hat{\tau}_1}), \end{cases}$$

$$\begin{cases} \hat{\tau}_k(\omega) = \inf\left\{t \geqq \hat{\tau}_{k-1}(\omega); X_t(\omega) + \sum_1^{k-1} \xi_i(\omega) \in D\right\} \\ \hat{\xi}_k(\omega) = \xi\left(X_{\hat{\tau}_k}(\omega) + \sum_1^{k-1} \hat{\xi}_i(\omega)\right), \quad k = 2, 3, 4, \ldots, \end{cases}$$

where $D = \{x; u(x) = \inf_{\xi \geqq 0} \{k(\xi) + u(x+\xi)\}\}$.

Remark. If $k(\xi)$ is continuous and tends to infinity as $|\xi|\to\infty$, then there exists a Borel function $\xi(x)$ satisfying the assumption in Theorem 2 because the infimum of the right-hand side of (0.9) is attained at some ξ.

§1. Proof of Theorem 1 (I) ($u(x)\leqq w(x)$)

For a $\mathscr{B}_t$-stopping time τ and R^n-valued random variable ξ, let θ_τ^ξ be the transformation of Ω defined by

$$X_t(\theta_\tau^\xi\omega)=X_{\tau(\omega)+t}(\omega)+\xi(\omega). \tag{1.1}$$

We note that if ξ is $\mathscr{B}_\tau$-measurable, then $(\theta_\tau^\xi)^{-1}\mathscr{B}_t\subset\mathscr{B}_{\tau+t}$ espeacially, if $\tau\equiv 0$ and ξ is $\mathscr{B}_0$-measurable, then $(\theta^\xi)^{-1}\mathscr{B}_t\equiv(\theta_0^\xi)^{-1}\mathscr{B}_t\subset\mathscr{B}_t$ (cf. Appendix). Further we note that if ξ is $\mathscr{B}_s$-measurable, then $(\theta^\xi)^{-1}\mathscr{F}_t\subset\mathscr{B}_t$ for each $t\geqq s$, where $\mathscr{F}_t=\sigma(X_u, u\leqq t)$.

Now we prepare three lemmas for the proof of inequality $u(x)\leqq w(x)$.

Lemma 1.1. *$u(x)$ is a bounded uniformly continuous function.*

Proof. (uniform continuity). By the definition of $u(x)$, for each $\varepsilon>0$ there exists a control $V(\varepsilon)=\{\tau_1^\varepsilon,\xi_1^\varepsilon;\tau_2^\varepsilon,\xi_2^\varepsilon;\ldots\}$ such that

$$u(x)\geqq E_x\left[\int_0^\infty e^{-\alpha t}f(Y_t^{v(\varepsilon)})\,dt+\sum_{i=1}^\infty e^{-\alpha\tau_i^\varepsilon}k(\xi_i)\right]-\varepsilon. \tag{1.2}$$

We define another control $\tilde v(\varepsilon)=\{\tilde\tau_1^\varepsilon,\tilde\xi_1^\varepsilon;\tilde\tau_2^\varepsilon,\tilde\xi_2^\varepsilon;\ldots\}$ as follows;

$$\tilde\tau_n^\varepsilon(\omega)=\tau_n^\varepsilon(\theta^\delta\omega)\qquad n=1,2,3,\ldots \tag{1.3}$$

$$\tilde\xi_n^\varepsilon(\omega)=\xi_n^\varepsilon(\theta^\delta\omega)\qquad n=1,2,3,\ldots \tag{1.4}$$

where $\delta\in R^n$. It is easily seen that $\tilde v(\varepsilon)\in V$ (cf. Appendix). Then

$$\begin{aligned}Y_s^{\tilde v(\varepsilon)}(\omega)&=X_s(\omega)+\sum_{\tilde\tau_i^\varepsilon(\omega)\leqq s}\tilde\xi_i^\varepsilon(\omega)\\&=X_s(\theta^\delta\omega)-\delta+\sum_{\tau_i^\varepsilon(\theta^\delta\omega)\leqq s}\xi_i^\varepsilon(\theta^\delta\omega)=Y_s^{v(\varepsilon)}(\theta^\delta\omega)-\delta\end{aligned}$$

and

$$\sum_i e^{-\alpha\tilde\tau_i^\varepsilon}k(\tilde\xi_i^\varepsilon)=\sum_i e^{-\alpha\tau_i^\varepsilon(\theta^\delta\omega)}k(\xi_i^\varepsilon(\theta^\delta\omega)).$$

Therefore for each $y\in R^n$

$$\left|E_y\left[\int_0^\infty e^{-\alpha t}f(Y_t^{\tilde v(\omega)})\,dt+\sum_i e^{-\alpha\tilde\tau_i^\varepsilon}k(\tilde\xi_i^\varepsilon)\right]\right.$$
$$\left.-E_{y+\delta}\left[\int_0^\infty e^{-\alpha t}f(Y_t^{v(\varepsilon)})\,dt+\sum_i e^{-\alpha\tau_i^\varepsilon}k(\xi_i^\varepsilon)\right]\right|$$

$$\leqq \left| E_y \left[\int_0^\infty e^{-\alpha t} \{ f(Y_t^{v(\varepsilon)}(\theta^\delta \omega) - \delta) - f(Y_t^{v(\varepsilon)}(\theta^\delta \omega)) \} \, dt \right] \right|$$
$$+ \left| E_y \left[\int_0^\infty e^{-\alpha t} f(Y_t^{v(\varepsilon)}(\theta^\delta \omega)) \, dt \right] - E_{y+\delta} \left[\int_0^\infty e^{-\alpha t} f(Y_t^{v(\varepsilon)}(\omega)) \, dt \right] \right|$$
$$+ \left| E_y \left[\sum_i e^{-\alpha \tau_i^\varepsilon(\theta^\delta \omega)} k(\xi_i^\varepsilon(\theta^\delta \omega)) \right] - E_{y+\delta} \left[\sum_i e^{-\alpha \tau_i^\varepsilon(\omega)} k(\xi_i^\varepsilon(\omega)) \right] \right|$$
$$\leqq \frac{1}{\alpha} \sup_z |f(z-\delta) - f(z)|.$$

Here we made use of spatial homogeneity of the process $(\Omega, \mathcal{B}, \mathcal{B}_t, X_t, P_x, \theta_t)$. Since $f(x)$ is uniformly continuous, there exists $\delta(\varepsilon) > 0$ such that

$$\left| E_y \left[\int_0^\infty e^{-\alpha t} f(Y_t^{\tilde{v}(\varepsilon)}) \, dt + \sum_i e^{-\alpha \tilde{\tau}_i^\varepsilon} k(\tilde{\xi}_i^\varepsilon) \right] \right.$$
$$\left. - E_{y+\eta} \left[\int_0^\infty e^{-\alpha t} f(Y_t^{v(\varepsilon)}) \, dt + \sum_i e^{-\alpha \tau_i^\varepsilon} k(\xi_i^\varepsilon) \right] \right| < \varepsilon$$

for each η with $|\eta| < \delta(\varepsilon)$. Take $y \in R^n$ such that $|x-y| < \delta(\varepsilon)$. We can assume that $u(x) \leqq u(y)$. Then

$$E_x \left[\int_0^\infty e^{-\alpha t} f(Y_t^{v(\varepsilon)}) \, dt + \sum_i e^{-\alpha \tau_i^\varepsilon} k(\xi_i^\varepsilon) \right] - \varepsilon \leqq u(x) \leqq u(y)$$
$$\leqq E_y \left[\int_0^\infty e^{-\alpha t} f(Y_t^{\tilde{v}(\varepsilon)}) \, dt + \sum_i e^{-\alpha \tilde{\tau}_i^\varepsilon} k(\tilde{\xi}_i^\varepsilon) \right].$$

Therefore

$$|u(x) - u(y)| < 2\varepsilon.$$

Boundedness of $u(x)$ is clear because the control such that $\tau_n \equiv \infty$ $(n = 1, 2, \ldots)$ is included in V and (A.1)$\sim$(A.3) are assumed.

Lemma 1.2. $u(x) \leqq E_x \left[\int_0^\tau e^{-\alpha t} f(X_t) \, dt + e^{-\alpha \tau} u(X_\tau) \right]$ *for each stopping time* τ.

Proof. For each $\delta > 0$ there exist sequences $\{x_k\}_k$, $x_k \in R^n$ and $\{U_{x_k}(\delta)\}_k$, where $U_{x_k}(\delta)$ is a neighborhood of x_k such that $\bigcup_k U_{x_k}(\delta) = R^n$, $|y - x_k| < \delta$ for each $y \in U_{x_k}(\delta)$ and $U_{x_k}(\delta) \cap U_{x_j}(\delta) = \phi$ if $k \neq j$. We can take a sequence of controls $\{v(n, \varepsilon)\}_n = \{\{\tau_1^{n,\varepsilon}, \xi_1^{n,\varepsilon}; \tau_2^{n,\varepsilon}, \ldots\}\}_n \subset V$ such that

$$E_{x_n} \left[\int_0^\infty e^{-\alpha t} f(Y_t^{v(n,\varepsilon)}) \, dt + \sum_i e^{-\alpha \tau_i^{n,\varepsilon}} k(\xi_i^{n,\varepsilon}) \right] - \varepsilon \leqq u(x_n),$$
$$n = 1, 2, \ldots, \text{for each } \varepsilon > 0.$$

Put

$$\tilde{\tau}_i^{n,\varepsilon}(\omega) = \tau_i^{n,\varepsilon}(\theta^{x_n - X_0(\omega)} \omega) \tag{1.5}$$
$$\tilde{\xi}_i^{n,\varepsilon}(\omega) = \xi_i^{n,\varepsilon}(\theta^{x_n - X_0(\omega)} \omega). \tag{1.6}$$

Then $\tilde{v}(n,\varepsilon)=\{\tilde{\tau}_1^{n,\varepsilon},\tilde{\zeta}_1^{n,\varepsilon},\tilde{\tau}_2^{n,\varepsilon},\tilde{\zeta}_2^{n,\varepsilon},\ldots\}\in V$ (see Appendix). Since $P_{x_n}(\tau_i^{n,\varepsilon}=\tilde{\tau}_i^{n,\varepsilon},\ \zeta_i^{n,\varepsilon}=\tilde{\zeta}_i^{n,\varepsilon}$ for all $i)=1$ it follows that

$$\left|E_x\left[\int_0^\infty e^{-\alpha t}f(Y_t^{\tilde{v}(n,\varepsilon)})\,dt+\sum_i e^{-\tilde{\tau}_i^{n,\varepsilon}}k(\tilde{\zeta}_i^{n,\varepsilon})\right]\right.$$
$$\left.-E_{x_n}\left[\int_0^\infty e^{-\alpha t}f(Y_t^{\tilde{v}(n,\varepsilon)})\,dt+\sum_i e^{-\alpha\tilde{\tau}_i^{n,\varepsilon}}k(\tilde{\zeta}_i^{n,\varepsilon})\right]\right|<\varepsilon$$

for each $x\in U_{x_n}(\delta)$, with $|\delta|<\delta(\varepsilon)$ in the same way as the proof of Lemma 1.1. Here $\delta(\varepsilon)$ is a positive number such that

$$\frac{1}{\alpha}|f(x)-f(y)|<\varepsilon \text{ holds whenever } |x-y|<\delta(\varepsilon).$$

Put

$$u^{n,\varepsilon}(x)=E_x\left[\int_0^\infty e^{-\alpha t}f(Y_t^{\tilde{v}(n,\varepsilon)})\,dt+\sum_i e^{-\alpha\tilde{\tau}_i^{n,\varepsilon}}k(\tilde{\zeta}_i^{n,\varepsilon})\right].$$

Then

$$|u(x_n)-u^{n,\varepsilon}(x_n)|<\varepsilon$$

and

$$|u^{n,\varepsilon}(x_n)-u^{n,\varepsilon}(y)|<\varepsilon \quad \text{for each } y\in U_{x_n}(\delta).$$

Moreover $|u(x_n)-u(y)|<\varepsilon$ by Lemma 1.1, and therefore for each stopping time τ,

$$\begin{aligned}
&E_x[e^{-\alpha\tau}u(X_\tau)]\\
&=\sum_{n=1}^\infty E_x[e^{-\alpha\tau}u(X_\tau);X_\tau\in U_{x_n}(\delta)]\\
&=\sum_{n=1}^\infty E_x[e^{-\alpha\tau}\{u(X_\tau)-u^{n,\varepsilon}(X_\tau)\};X_\tau\in U_{x_n}(\delta)]\\
&\quad+\sum_{n=1}^\infty E_x[e^{-\alpha\tau}u^{n,\varepsilon}(X_\tau);X_\tau\in U_{x_n}(\delta)]\\
&\geqq\sum_{n=1}^\infty E_x[e^{-\alpha\tau}u^{n,\varepsilon}(X_\tau);X_\tau\in U_{x_n}(\delta)]-3\varepsilon\\
&=\sum_{n=1}^\infty E_x\left[e^{-\alpha\tau}E_{X_\tau}\left[\int_0^\infty e^{-\alpha s}f(Y_s^{\tilde{v}(n,\varepsilon)})\,ds\right.\right.\\
&\quad\left.\left.+\sum_i e^{-\alpha\tilde{\tau}_i^{n,\varepsilon}}k(\tilde{\zeta}_i^{n,\varepsilon})\right];X_\tau\in U_{x_n}(\delta)\right]-3\varepsilon.
\end{aligned}$$

Let $v(\varepsilon)=\{\tau_1^\varepsilon,\zeta_1^\varepsilon;\tau_2^\varepsilon,\zeta_2^\varepsilon;\ldots\}$ be a control such that

$$\tau_k^\varepsilon(\omega)=\tau(\omega)+\tilde{\tau}_k^{n,\varepsilon}(\theta_\tau\omega) \quad \text{if } X_\tau(\omega)\in U_{x_n}(\delta), \tag{1.7}$$

and

$$\zeta_k^\varepsilon(\omega)=\tilde{\zeta}_k^{n,\varepsilon}(\theta_\tau\omega) \quad \text{if } X_\tau(\omega)\in U_{x_n}(\delta), \tag{1.8}$$

$k=1,2,\ldots;\ n=1,2,\ldots$. Then $v(\varepsilon)\in V$ (see Appendix). When $X_\tau(\omega)\in U_{x_n}(\delta)$,

$$\begin{aligned} Y_s^{v(\varepsilon)}(\omega) &= X_s(\omega)+\sum_{\tau_i^\varepsilon(\omega)\leqq s}\xi_i^\varepsilon(\omega) \\ &= X_s(\omega)+\sum_{\tau(\omega)+\tilde{\tau}_i^{n,\varepsilon}(\theta_\tau\omega)\leqq s}\tilde{\xi}_i^{n,\varepsilon}(\theta_\tau\omega) \\ &= X_{s-\tau}(\theta_\tau\omega)+\sum_{\tilde{\tau}_i^{n,\varepsilon}(\theta_\tau\omega)\leqq s-\tau}\tilde{\xi}_i^{n,\varepsilon}(\theta_\tau\omega) \\ &= Y_{s-\tau}^{\tilde{v}(n,\varepsilon)}(\theta_\tau\omega) \end{aligned}$$

for each $s\geqq\tau$ and

$$\begin{aligned} \sum_i e^{-\alpha\tau_i^\varepsilon}k(\xi_i) &= \sum_i e^{-\alpha(\tau+\tilde{\tau}_i^{n,\varepsilon}(\theta_\tau\omega))}k(\tilde{\xi}_i^{n,\varepsilon}(\theta_\tau\omega)) \\ &= e^{-\alpha\tau}\sum_i e^{-\alpha\tilde{\tau}_i^{n,\varepsilon}(\theta_\tau\omega)}k(\tilde{\xi}_i^{n,\varepsilon}(\theta_\tau\omega)). \end{aligned}$$

Therefore

$$\begin{aligned} &E_x\left[\int_0^\tau e^{-\alpha t}f(X_t)\,dt+e^{-\alpha\tau}u(X_\tau)\right] \\ &\geqq E_x\left[\int_0^\tau e^{-\alpha t}f(Y_t^{v(\varepsilon)})\,dt\right] \\ &\quad+\sum_{n=1}^\infty E_x\left[e^{-\alpha\tau}E_{X_\tau}\left[\int_0^\infty e^{-\alpha s}f(Y_s^{\tilde{v}(n,\varepsilon)})\,ds+\sum_i e^{-\alpha\tilde{\tau}_i^{n,\varepsilon}}k(\tilde{\xi}_i^{n,\varepsilon})\right];\ X_\tau\in U_{x_n}(\delta)\right]-3\varepsilon \\ &= E_x\left[\int_0^\tau e^{-\alpha t}f(Y_t^{v(\varepsilon)})\,dt\right] \\ &\quad+\sum_{n=1}^\infty E_x\left[\int_0^\infty e^{-\alpha(\tau+s)}f(Y_s^{\tilde{v}(n,\varepsilon)}(\theta_\tau\omega))\,ds\right. \\ &\quad\left.+\sum_i e^{-\alpha(\tau(\omega)+\tilde{\tau}_i^{n,\varepsilon}(\theta_\tau\omega))}k(\tilde{\xi}_i^{n,\varepsilon}(\theta_\tau\omega));\ X_\tau\in U_{x_n}(\delta)\right]-3\varepsilon \\ &= E_x\left[\int_0^\tau e^{-\alpha t}f(Y_t^{v(\varepsilon)})\,dt\right]+E_x\left[\int_0^\infty e^{-\alpha(\tau+s)}f(Y_{\tau+s}^{v(\varepsilon)})\,ds+\sum_i e^{-\alpha\tau_i^\varepsilon}k(\xi_i^\varepsilon)\right]-3\varepsilon \\ &= E_x\left[\int_0^\infty e^{-\alpha t}f(Y_t^{v(\varepsilon)})\,dt+\sum_i e^{-\alpha\tau_i^\varepsilon}k(\xi_i^\varepsilon)\right]-3\varepsilon \\ &\geqq u(x)-3\varepsilon. \end{aligned}$$

Since ε is arbitrary our lemma has been proved.

Lemma 1.3. $u(x)\leqq\inf_{z\geqq 0}\{k(z)+u(x+z)\}$.

Proof. Let $v\in V$ be an arbitrary control and define another control $\tilde{v}=\{\tilde{\tau}_1,\tilde{\xi}_1;\tilde{\tau}_2,\tilde{\xi}_2;\ldots\}$ as follows:

$$\begin{aligned} &\tilde{\tau}_1(\omega)\equiv 0, \quad \tilde{\xi}_1(\omega)\equiv z \\ &\tilde{\tau}_i(\omega)=\tau_{i-1}(\theta^z\omega), \quad \tilde{\xi}_i(\omega)=\xi_{i-1}(\theta^z\omega), \quad i=2,3,4,\ldots \end{aligned}$$

where $z \in [0, \infty)^n$. Then

$$Y_s^{\tilde{v}}(\omega) = X_s(\omega) + z + \sum_{\tau_i(\theta^z \omega) \leqq s} \xi_i(\theta^z \omega) = Y_s^v(\theta^z \omega).$$

Therefore

$$\begin{aligned} & E_x\left[\int_0^\infty e^{-\alpha s} f(Y_s^{\tilde{v}})\, ds + \sum_i e^{-\alpha \tau_i} k(\tilde{\xi}_i)\right] \\ & = k(z) + E_x\left[\int_0^\infty e^{-\alpha s} f(Y_s^v(\theta^z \omega))\, ds + \sum_i e^{-\alpha \tau_i(\theta^z \omega)} k(\xi_i(\theta^z \omega))\right] \\ & = k(z) + E_{x+z}\left[\int_0^\infty e^{-\alpha s} f(Y_s^v)\, ds + \sum_i e^{-\alpha \tau_i} k(\xi_i)\right]. \end{aligned}$$

So we have

$$u(x) \leqq k(z) + E_{x+z}\left[\int_0^\infty e^{-\alpha s} f(Y_s^v)\, ds + \sum_i e^{-\alpha \tau_i} k(\xi_i)\right].$$

Since the left hand side does not depend on z and v it hold that

$$u(x) \leqq \inf_{z \geqq 0} \{k(z) + u(x+z)\}.$$

Now we can easily prove that $u(x) \leqq w(x)$ as the consequence of Lemma 1.2 and Lemma 1.3. We have

$$\begin{aligned} u(x) & \leqq E_x\left[\int_0^\tau e^{-\alpha t} f(X_t)\, dt + e^{-\alpha \tau} u(X_\tau)\right] \\ & \leqq E_x\left[\int_0^\tau e^{-\alpha t} f(X_t)\, dt + e^{-\alpha \tau} M_u(X_\tau)\right] \end{aligned}$$

for each stopping time τ. So we have $u(x) \leqq w(x)$.

§ 2. Some Lemmas

In this section we prepare some lemmas to complete the proof of Theorem 1 and Theorem 2.

Lemma 2.1. *$M_u(x)$ is a bounded uniformly continuous function.*

Proof. For each $\varepsilon > 0$ we can take $\delta(\varepsilon)$ such that

$$|u(x+z) - u(y+z)| < \varepsilon$$

for $|x-y| < \delta(\varepsilon)$ and for each z by uniform continuity of $u(x)$. We can assume that

$$M_u(y) \leqq M_u(x).$$

By definition of $M_u(y)$, there exists ξ_ε such that

$$M_u(y) \geqq k(\xi_\varepsilon) + u(y+\xi_\varepsilon) - \varepsilon$$

for each $\varepsilon > 0$. Therefore

$$|k(\xi_\varepsilon) + u(x+\xi_\varepsilon) - \{k(\xi_\varepsilon) + u(y+\xi_\varepsilon)\}| \leqq |u(x+\xi_\varepsilon) - u(y+\xi_\varepsilon)| < \varepsilon$$

for $|x-y| < \delta(\varepsilon)$. Hence

$$|M_u(x) - M_u(y)| < 2\varepsilon \quad \text{for } |x-y| < \delta(\varepsilon).$$

Boundedness of $M_u(x)$ is clear.

Lemma 2.2. *$G_\alpha f(x) - w(x)$ is an α-excessive function and $w(x)$ can be written*

$$w(x) = E_x\left[\int_0^{\tau_D} e^{-\alpha t} f(X_t)\, dt + e^{-\alpha \tau_D} M_u(X_{\tau_D})\right] \tag{2.1}$$

where $\tau_D = \inf\{t \geqq 0;\ X_t \in D\}$, $D = \{x;\ w(x) = M_u(x)\}$.

Proof. By the definition of $w(x)$ and the strong Markov property,

$$G_\alpha f(x) - w(x) = \sup_\tau E_x[e^{-\alpha\tau}\{G_\alpha f(X_\tau) - M_u(X_\tau)\}].$$

We note that our process $(\Omega, \mathscr{B}, \mathscr{B}_t, P_x, X_t, \theta_t)$ is Hunt and has a Feller transition semi-group. So, by [3] (III. § 3 Theorem 1) it is known that $G_\alpha f(x) - w(x)$ is the smallest excessive majorant of $G_\alpha f(x) - M_u(x)$. Moreover, since $G_\alpha f(x) - M_u(x)$ is bounded continuous we can easily show (2.1) by making use of Lemma 10 ([3], III § 4) and Lemma 3 ([3], III § 2).

Lemma 2.3.

$$w(x) \leqq \inf_{\xi \geqq 0} \{k(\xi) + w(x+\xi)\}. \tag{2.2}$$

Proof. It is easily seen that (2.2) follows from the definition of $w(x)$ and the inequality $u(x) \leqq w(x)$:

$$w(x) \leqq M_u(x) = \inf_{\xi \geqq 0} \{k(\xi) + u(x+\xi)\} \leqq \inf_{\xi \geqq 0} \{k(\xi) + w(x+\xi)\}.$$

Lemma 2.4. *Let g be an $\mathscr{F}$-measurable random variable*[1], *τ be a stopping time and ξ be a $\mathscr{B}_\tau$-measurable R^n valued random vector. Then*

$$E_x[g(\theta_\tau^\xi \omega); B] = E_x[E_{X_\tau + \xi}[g(\omega)]; B] \tag{2.3}$$

for each $B \in \mathscr{B}_\tau$.

Proof. Noting that our process has a Feller transition semi-group, we have only to prove (2.3) in case that

$$g(\omega) = g_1(X_{t_1}(\omega))\, g_2(X_{t_2}(\omega)) \ldots g_m(X_{t_m}(\omega)),$$
$$0 \leqq t_1 < t_2 < \ldots < t_m, \quad g_k \in C_0(R^n), \quad k = 1, 2, \ldots, m$$

---

[1] Here we prove this lemma for an $\mathscr{F}$-measurable r.v. g, but we can extend it to a $\mathscr{B}$-measurable r.v. g (cf. [2] Chap. 1 § 5).

and

$$\xi(\omega)=\sum_{k=1}^{N}\xi_k I_{\Gamma_k}(\omega)$$

where $\Gamma_k\in\mathscr{B}_\tau$ are disjoint and $\xi_k\in R^n$ are constant vectors. $\tau_n=\dfrac{[2^n\tau]+1}{2^n}$. Then $\tau_n\downarrow\tau$ as $n\to\infty$, and

$$\begin{aligned}
&E_x[g(\theta_\tau^\xi\omega);B]\\
&\quad=E_x[g_1(X_{t_1+\tau}+\xi)\dots g_m(X_{t_m+\tau}+\xi);B]\\
&\quad=\lim_{n\to\infty}E_x[g_1(X_{t_1+\tau_n}+\xi)\dots g_m(X_{t_m+\tau_n}+\xi);B]\\
&\quad=\lim_{n\to\infty}\sum_{j=1}^{\infty}E_x[g_1(X_{t_1+\frac{j}{2^n}}+\xi)\dots g_m(X_{t_m+\frac{j}{2^n}}+\xi);\\
&\qquad B\cap\left\{\frac{j-1}{2^n}\leqq\tau<\frac{j}{2^n}\right\}]\\
&\quad=\lim_{n\to\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{N}E_x[g_1(X_{t_1+\frac{j}{2^n}}+\xi_k)\dots g_m(X_{t_m+\frac{j}{2^n}}+\xi_k);\\
&\qquad B\cap\Gamma_k\cap\left\{\frac{j-1}{2^n}\leqq\tau<\frac{j}{2^n}\right\}]\\
&\quad=\lim_{n\to\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{N}E_x[E_{X_{\frac{j}{2^n}+\xi_k}}[g_1(X_{t_1})\dots g_m(X_{t_m})];\\
&\qquad B\cap\Gamma_k\cap\left\{\frac{j-1}{2^n}\leqq\tau<\frac{j}{2^n}\right\}]\\
&\quad=\lim_{n\to\infty}\sum_{j=1}^{\infty}E_x[E_{X_{\frac{j}{2^n}+\xi}}[g_1(X_{t_1})\dots g_m(X_{t_m})];\\
&\qquad B\cap\left\{\frac{j-1}{2^n}\leqq\tau<\frac{j}{2^n}\right\}]\\
&\quad=\lim_{n\to\infty}E_x[E_{X_{\tau_n+\xi}}[g_1(X_{t_1})\dots g_m(X_{t_m})];B]\\
&\quad=E_x[E_{X_\tau+\xi}[g(\omega)];B],
\end{aligned}$$

because our process is spatially homogeneous and has a Feller transition semigroup.

Lemma 2.5. *$Q(x)=G_\alpha f(x)-w(x)$ is a regular α-potential (cf. [2]).*

Proof. Since it is known by Lemma 2.2 that $G_\alpha f(x)-w(x)$ is α-excessive, we have only to prove

$$\lim_{n\to\infty}E_x[e^{-\alpha T_n}Q(X_{T_n})]=E_x[e^{-\alpha T}Q(X_T)]$$

where $\{T_n\}$ is any sequence of stopping times such that $T_n \uparrow T$ as $n \to \infty$. By Lemma 2.2 and the strong Markov property we have

$$\begin{aligned} Q(x) &= G_\alpha f(x) - w(x) \\ &= E_x[e^{-\alpha\tau_D}\{G_\alpha f(X_{\tau_D}) - M_u(X_{\tau_D})\}] \equiv E_x[e^{-\alpha\tau_D} K(X_{\tau_D})]. \end{aligned}$$

Therefore

$$\begin{aligned} E_x[e^{-\alpha T_n} Q(X_{T_n})] &= E_x[e^{-\alpha T_n} E_{X_{T_n}}[e^{-\alpha\tau_D} K(X_{\tau_D})]] \\ &= E_x[e^{-\alpha(T_n + \tau_D(\theta_{T_n}\omega))} K(X_{T_n + \tau_D(\theta_{T_n}\omega)}(\omega)]. \end{aligned}$$

Since

$$\begin{aligned} T_n + \tau_D(\theta_{T_n}\omega) &= \inf\{t \geqq T_n;\, X_t \in D\} \uparrow \inf\{t \geqq T;\, X_t \in D\} \\ &= T + \tau_D(\theta_T \omega), \end{aligned}$$

it holds that

$$\begin{aligned} &\lim_{n\to\infty} E_x[e^{-\alpha(T_n + \tau_D(\theta_{T_n}\omega))} K(X_{T_n + \tau_D(\theta_{T_n}\omega)}(\omega))] \\ &= E_x[e^{-\alpha(T + \tau_D(\theta_T\omega))} K(X_{T + \tau_D(\theta_T\omega)}(\omega))] \\ &= E_x[e^{-\alpha T} E_{X_T}[e^{-\alpha\tau_D} K(X_{\tau_D})]] = E_x[e^{-\alpha T} Q(X_T)]. \end{aligned}$$

Here we made use of boundedness and continuity of $K(x)$ and the quasi-left continuity of our process.

§ 3. Proof of Theorem 1 (II) ($w(x) \leqq u(x)$),

Now we can complete the proof of Theorem 1.

Since $G_\alpha f(x) - w(x)$ is a regular α-potential by Lemma 2.5, there exists a non-negative continuous additive functional[2] such that

$$G_\alpha f(x) - w(x) = E_x\left[\int_0^\infty e^{-\alpha t}\, dA_t\right]$$

(cf. [2]). Put

$$\eta_i(\omega) = \sum_{j=1}^{i} \xi_j(\omega),$$

then

$$\begin{aligned} &E_x[e^{-\alpha\tau_i} w(Y^v(\tau_i -))] \\ &= E_x\left[e^{-\alpha\tau_i}\left\{E_{Y^{\hat{v}}(\tau_i -)}\left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt\right]\right.\right. \\ &\quad \left.\left. - E_{Y^{\hat{v}}(\tau_i -)}\left[\int_0^\infty e^{-\alpha t}\, dA_t\right]\right\}\right] \end{aligned}$$

[2] This additive functional is perfect by the assumption that our process has a reference measure (cf. [2] Chap. V Theorem 2.1).

$$
\begin{aligned}
&= E_x\left[e^{-\alpha\tau_i} E_{X(\tau_i)+\eta_{i-1}}\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right]\right] \\
&\quad - E_x\left[e^{-\alpha\tau_i} E_{X(\tau_i)+\eta_{i-1}}\left[\int_0^\infty e^{-\alpha t}\,dA_t\right]\right] \\
&= E_x\left[e^{-\alpha\tau_i}\int_0^\infty e^{-\alpha t} f(X_t(\theta^{\eta_{i-1}}_{\tau_i}\omega))\,dt\right] \\
&\quad - E_x\left[e^{-\alpha\tau_i}\int_0^\infty e^{-\alpha t}\,dA_t(\theta^{\eta_{i-1}}_{\tau_i}\omega)\right] \\
&= E_x\left[\int_{\tau_i}^\infty e^{-\alpha t} f(X_t(\theta^{\eta_{i-1}}\omega))\,dt\right] - E_x\left[\int_{\tau_i-\tau_{i-1}}^\infty e^{-\alpha t}\,dA_t(\theta^{\eta_{i-1}}_{\tau_{i-1}}\omega)\right].
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&E_x[e^{-\alpha\tau_{i-1}} w(Y^v(\tau_{i-1}))] \\
&\quad = E_x\left[\int_{\tau_{i-1}}^\infty e^{-\alpha t} f(X_t(\theta^{\eta_{i-1}}\omega))\,dt - \int_0^\infty e^{-\alpha t}\,dA_t(\theta^{\eta_{i-1}}_{\tau_{i-1}}\omega)\right]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&E_x[e^{-\alpha\tau_i} w(Y^v(\tau_i-))] - E_x[e^{-\alpha\tau_{i-1}} w(Y^v(\tau_{i-1}))] \\
&\quad = E_x\left[-\int_{\tau_{i-1}}^{\tau_i} e^{-\alpha t} f(X_t(\theta^{\eta_{i-1}}\omega))\,dt + \int_0^{\tau_i-\tau_{i-1}} e^{-\alpha t}\,dA_t(\theta^{\eta_{i-1}}_{\tau_{i-1}}\omega)\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&E_x[e^{-\alpha\tau_i} w(Y^v(\tau_i-)) - e^{-\alpha\tau_{i-1}} w(Y^v(\tau_{i-1}))] \\
&\quad \geqq -E_x\left[\int_{\tau_{i-1}}^{\tau_i} e^{-\alpha t} f(X_t(\theta^{\eta_{i-1}}\omega))\,dt\right] = -E_x\left[\int_{\tau_{i-1}}^{\tau_i} e^{-\alpha t} f(Y_t^v)\,dt\right].
\end{aligned}
$$

By Blumenthal's $0-1$ law we know that $P_x(\tau_1=0)=0$ or 1. In the case that $P_x(\tau_1=0)=0$, putting $\tau_0\equiv 0$ we have

$$
\begin{aligned}
&\sum_{i=1}^n \{E_x[e^{-\alpha\tau_i} w(Y^v(\tau_i-))] - E_x[e^{-\alpha\tau_{i-1}} w(Y^v(\tau_{i-1}))]\} \\
&\quad \geqq -E_x\left[\int_0^{\tau_n} e^{-\alpha t} f(Y_t^v)\,dt\right].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&-E_x[w(Y_0^v)] + \sum_{i=1}^{n-1} E_x[e^{-\alpha\tau_i}\{w(Y^v(\tau_i-)) - w(Y^v(\tau_i))\}] + E_x[e^{-\alpha\tau_n} w(Y^v(\tau_n-))] \\
&\quad \geqq -E_x\left[\int_0^{\tau_n} e^{-\alpha t} f(Y_t^v)\,dt\right].
\end{aligned}
$$

On the other hand we know by Lemma 2.3

$$w(Y^v(\tau_i-)) \leqq \inf_{z \geqq 0} \{k(z)+w(Y^v(\tau_i-)+z)\}$$
$$\leqq k(\xi_i)+w(Y^v(\tau_i))$$

and

$$E_x[w(Y_0^v)]=w(x)$$

by our assumption. So we have

$$w(x) \leqq E_x\left[\int_0^{\tau_n} e^{-\alpha t} f(Y_t^v)\,dt+\sum_{i=1}^{\infty} e^{-\alpha\tau_i} k(\xi_i)\right]+E_x[e^{-\alpha\tau_n} w(Y^v(\tau_n-))].$$

Since $\tau_n \to \infty$ as $n \to \infty$, it holds that

$$w(x) \leqq E_x\left[\int_0^{\infty} e^{-\alpha t} f(Y_t^v)\,dt+\sum_{i=1}^{\infty} e^{-\alpha\tau_i} k(\xi_i)\right]$$

for each $v \in V$. Hence we obtain $w(x) \leqq u(x)$.

In the other case that $P_x(\tau_1=0)=1$ we have

$$E_x[w(Y^v(\tau_1-))]=E_x[w(X_0)]=w(x).$$

Therefore it holds that

$$E_x[w(Y^v(\tau_1-))-w(Y^v(\tau_1))]+\sum_{i=2}^{n-1} E_x[e^{-\alpha\tau_i}\{w(Y^v(\tau_i-))-w(Y^v(\tau_i))\}]$$
$$\geqq w(x)-E_x\left[\int_0^{\tau_n} e^{-\alpha t} f(Y_t^v)\,dt\right]-E_x[e^{-\alpha\tau_n} w(Y^v(\tau_n-))].$$

We can then obtain $w(x) \leqq u(x)$ in the same way as above.

Remark. Put $E=\{g(x); G_\alpha f(x)-g(x)$ is a bounded α-regular potential and $g(x) \leqq M_g(x)\}$. Then by the above proof of the inequality $w(x) \leqq u(x)$ we can see $u(x)$ is the largest of functions belonging to E.

§ 4. Proof of Theorem 2

First we will prove that

$$u(Y^{\hat{v}}(\hat{\tau}_n)) \leqq M_u(Y^{\hat{v}}(\hat{\tau}_n))-c \quad \text{for each } n \tag{4.1}$$

where c is the constant in the assumption (A.3). From the equality

$$u(Y^{\hat{v}}(\hat{\tau}_n))+k(\hat{\xi}_n)=\inf_{z \geqq 0}\{u(Y^{\hat{v}}(\hat{\tau}_n-)+z)+k(z)\}$$

it follows that

$$u(Y^{\hat{v}}(\hat{\tau}_n)) \leqq u(Y^{\hat{v}}(\hat{\tau}_n-)+\hat{\xi}_n+z)+k(\hat{\xi}_n+z)-k(\hat{\xi}_n)$$
$$\leqq u(Y^{\hat{v}}(\hat{\tau}_n)+z)+k(z)-c$$

for each $z \in [0,\infty)^n$.

Therefore (4.1) holds and we know

$$Y^{\theta}(\hat{\tau}_n)\in D^c$$

for each n.

As the consequences of (4.1) we can prove the next lemma.

Lemma 4.1. $P_x(\lim_{n\to\infty}\hat{\tau}_n=\infty)=1$ *for each* n.

Proof. For each $\varepsilon>0$, there exists $\delta>0$ such that

$$|u(x)-u(y)|<\varepsilon \quad \text{and} \quad |M_u(x)-M_u(y)|<\varepsilon \tag{4.2}$$

hold whenever $|x-y|<\delta$. Then from (4.1) it follows that

$$u(Y^{\theta}(\hat{\tau}_n)+x)-\varepsilon\leqq M_u(Y^{\theta}(\hat{\tau}_n)+x)+\varepsilon-c, \qquad \text{for } |x|<\delta.$$

Take $\varepsilon<\dfrac{c}{2}$ and fix δ satisfying (4.2) for such ε. Then

$$u(Y^{\theta}(\hat{\tau}_n)+x)<M_u(Y^{\theta}(\hat{\tau}_n)+x) \qquad \text{for } |x|<\delta.$$

Therefore

$$Y^{\theta}(\hat{\tau}_n)+x\in D^c \qquad \text{for } |x|<\delta.$$

Now we define a stopping time $\sigma=\inf\{t\geqq 0;\ |X_t-X_0|>\delta\}$. Then

$$\begin{aligned}\hat{\tau}_{n+1}(\omega)&=\hat{\tau}_n(\omega)+\tau_D(\theta_{\hat{\tau}_n(\omega)}\theta^{\beta_n(\omega)}\omega)\\&\geqq\hat{\tau}_n(\omega)+\sigma(\theta_{\hat{\tau}_n}\omega).\end{aligned}$$

Hence

$$\begin{aligned}E_x[e^{-\hat{\tau}_n}]&\leqq E_x[e^{-\hat{\tau}_{n-1}-\sigma(\theta_{\hat{\tau}_{n-1}}\omega)}]\\&=E_x[e^{-\hat{\tau}_{n-1}}E_{X_{\hat{\tau}_{n-1}}}[e^{-\sigma}]].\end{aligned}$$

Since $E_x[e^{-\sigma}]\equiv h$ is a constant we obtain

$$E_x[e^{-\hat{\tau}_n}]<h^n.$$

Then

$$\lim_{n\to\infty}E_x[e^{-\hat{\tau}_n}]=0.$$

Therefore

$$P_x(\lim_{n\to\infty}\hat{\tau}_n=\infty)=1.$$

Now we can give the proof of Theorem 2.

Since $\tau_D(\omega)=\tau(\omega)+\tau_D(\theta_\tau\omega)$ for each stopping time $\tau\leqq\tau_D$,

$$\begin{aligned}w(x)&=E_x\left[\int_0^{\tau(\omega)+\tau_D(\theta_\tau\omega)}e^{-\alpha t}f(X_t)\,dt+e^{-\alpha(\tau(\omega)+\tau_D(\theta_\tau\omega))}M_u(X_{\tau_D(\theta_\tau\omega)}(\theta_\tau\omega))\right]\\&=E_x\left[\int_0^{\tau}e^{-\alpha t}f(X_t)\,dt+e^{-\alpha\tau}\,E_{X_\tau}\left[\int_0^{\tau_D}e^{-\alpha t}f(X_t)\,dt+e^{-\alpha\tau_D}M_u(X_{\tau_D})\right]\right]\\&=E_x\left[\int_0^{\tau}e^{-\alpha t}f(X_t)\,dt+e^{-\alpha\tau}w(X_\tau)\right].\end{aligned}$$

Putting $\tau=\tau_D$,

$$u(x)=E_x\left[\int_0^{\tau_D} e^{-\alpha t}f(X_t)\,dt+e^{-\alpha\tau_D}u(X_{\tau_D})\right], \tag{6.2}$$

because $u(x)=w(x)$. Therefore

$$u(x)=E_x\left[\int_0^{\hat\tau_1} e^{-\alpha t}f(Y_t^{\hat\theta})\,dt+e^{-\alpha\hat\tau_1}u(Y^{\hat\theta}(\hat\tau_1-))\right] \tag{6.3}$$

On the other hand $X_{\hat\tau_1}=Y^{\hat\theta}(\hat\tau_1-)\in D$ because D is a closed set. Therefore by our assumption

$$\begin{aligned} u(Y^{\hat\theta}(\hat\tau_1-))&=k(\xi(Y^{\hat\theta}(\hat\tau_1-))+u(Y^{\hat\theta}(\hat\tau_1-)+\xi(Y^{\hat\theta}(\hat\tau_1-)) \\ &=k(\hat\xi_1)+u(Y^{\hat\theta}(\hat\tau_1)). \end{aligned} \tag{6.4}$$

Hence

$$\begin{aligned} u(x)=E_x&\left[\int_0^{\hat\tau_1} e^{-\alpha t}f(Y_t^{\hat\theta})\,dt+e^{-\alpha\hat\tau_1}k(\hat\xi_1)\right] \\ &+E_x[e^{-\alpha\hat\tau_1}u(Y^{\hat\theta}(\hat\tau_1))]. \end{aligned} \tag{6.5}$$

Now

$$\begin{aligned} &E_x[e^{-\alpha\hat\tau_n}u(Y^{\hat\theta}(\hat\tau_n))] \\ &=E_x\left[e^{-\alpha\hat\tau_n}E_{Y^{\hat\theta}(\hat\tau_n)}\left[\int_0^{\tau_D} e^{-\alpha t}f(X_t)\,dt+e^{-\alpha\tau_D}u(X_{\tau_D})\right]\right] \\ &=E_x\left[e^{-\alpha\hat\tau_n}E_{X_{\hat\tau_n+\hat\eta_n}}\left[\int_0^{\tau_D} e^{-\alpha t}f(X_t)\,dt+e^{-\alpha\tau_D}u(X_{\tau_D})\right]\right] \\ &=E_x\left[e^{-\alpha\hat\tau_n}\left\{\int_0^{\tau_D(\theta^{\hat\eta_n}_{\hat\tau_n}\omega)} e^{-\alpha t}f(X_t(\theta^{\hat\eta_n}_{\hat\tau_n}\omega))\,dt+e^{-\alpha\tau_D(\theta^{\hat\eta_n}_{\hat\tau_n}\omega)}u(X_{\tau_D(\theta^{\hat\eta_n}_{\hat\tau_n}\omega)}(\theta^{\hat\eta_n}_{\hat\tau_n}\omega))\right\}\right] \\ &=E_x\left[\int_{\hat\tau_n}^{\hat\tau_{n+1}} e^{-\alpha t}f(Y_t^{\hat\theta})\,dt+e^{-\alpha\hat\tau_{n+1}}u(Y^{\hat\theta}(\hat\tau_{n+1}-))\right] \end{aligned} \tag{6.6}$$

for each n. Then

$$u(x)=E_x\left[\int_0^{\hat\tau_2} e^{-\alpha t}f(Y_t^{\hat\theta})\,dt+e^{-\alpha\hat\tau_1}k(\hat\xi_1)+e^{-\alpha\hat\tau_2}u(Y^{\hat\theta}(\hat\tau_2-))\right]$$

by virtue of (6.6) for $n=1$. Hence we obtain

$$u(x)=E_x\left[\int_0^{\hat\tau_n} e^{-\alpha t}f(Y_t^{\hat\theta})\,dt+\sum_{i=1}^{n} e^{-\alpha\hat\tau_i}k(\hat\xi_i)+e^{-\alpha\hat\tau_n}u(Y^{\hat\theta}(\hat\tau_n))\right]$$

by induction. Lemma 4.1 now completes the proof.

Appendix

Let $\mathscr{P}$ be the family of all probability measures on R^n. We denote by $\mathscr{F}^\mu$ (resp. $\mathscr{F}_t^\mu$) the completion of $\mathscr{F}$ (resp. completion of $\mathscr{F}_t$ in $\mathscr{F}^\mu$) with respect to P_μ, $\mu\in\mathscr{P}$,

where $\mathscr{F}=\sigma(X_t, 0\leqq t)$ and $\mathscr{F}_t=\sigma(X_s, s\leqq t)$. Our process $(\Omega, \mathscr{B}, \mathscr{B}_t, P_x, X_t, \theta_t)$ is of function space type, that is $\Omega=D([0,\infty)\rightsquigarrow R^n)$, $\mathscr{B}=\bigcap_{\mu\in\mathscr{P}}\mathscr{F}^\mu$ and $\mathscr{B}_t=\bigcap_{\mu\in\mathscr{P}}\mathscr{F}_t^\mu$, where $D([0,\infty)\rightsquigarrow R^n)$ is the space of all right continuous functions on $[0,\infty)$ with lefthand limits.

We will derive a lemma concerning measurability of the transformation θ_τ^ξ defined by (1.1).

Lemma. *Let τ be a $\mathscr{B}_t$-stopping time and ξ be a $\mathscr{B}_\tau$-measurable random vector. Then $(\theta_\tau^\xi)^{-1}\mathscr{B}_s\subset\mathscr{B}_{\tau+s}$ for each $s\geqq 0$.*

Proof. First we note that $(\theta_\tau^\xi)^{-1}\mathscr{F}_s\subset\mathscr{B}_{\tau+s}$. Indeed $\{B\in\mathscr{F}_s;\ (\theta_\tau^\xi)^{-1}B\in\mathscr{B}_{\tau+s}\}$ is a σ-algebra and contains any cylinder set belonging to $\mathscr{F}_s$. Therefore we have only to prove that for any probability measure $\mu\in\mathscr{P}$ there exists another probability measure $\nu\in\mathscr{P}$ such that $P_\mu\circ(\theta_\tau^\xi)^{-1}=P_\nu$ according to Lemma 5.6 of Chapter I in [2]. It follows from Lemma 2.4 that

$$\begin{aligned}P_\mu((\theta_\tau^\xi)^{-1}B)&=E_\mu[I_B(\theta_\tau^\xi\omega)]\\&=E_\mu[E_{X_\tau+\xi}[I_B(\omega)]]\\&=\int P_y(B)\,P_\mu(X_\tau+\xi\in dy)\qquad\text{for each } B\in\mathscr{F}_s.\end{aligned}$$

Put $\nu(dy)=P_\mu(X_\tau+\xi\in dy)$. Then

$$P_\mu((\theta_\tau^\xi)^{-1}B)=P_\nu(B).\quad\text{q.e.d.}$$

Now we will make some remarks.

Let $v=\{\tau_1,\xi_1;\tau_2,\xi_2;\ldots\}$ be an impulsive control and η be a $\mathscr{B}_0$-measurable random vector. Put

$$\begin{aligned}\tilde{\tau}_i(\omega)&=\tau_i(\theta^\eta\omega)\\\tilde{\xi}_i(\omega)&=\xi_i(\theta^\eta\omega).\end{aligned}$$

Then $\tilde{v}=\{\tilde{\tau}_1,\tilde{\xi}_1;\tilde{\tau}_2,\tilde{\xi}_2;\ldots\}$ is also an impulsive control. For it holds that

$$\{\omega;\tilde{\tau}_i(\omega)<t\}=(\theta^\eta)^{-1}\{\tau_i(\omega)<t\}\in\mathscr{B}_t,\ t>0$$

and

$$\begin{aligned}&\{\omega;\tilde{\xi}_i(\omega)\in B\}\cap\{\omega;\tilde{\tau}_i(\omega)<t\}\\&\quad=[(\theta^\eta)^{-1}\{\xi_i(\omega)\in B\}]\cap[(\theta^\eta)^{-1}\{\tau_i(\omega)<t\}]\\&\quad=(\theta^\eta)^{-1}[\{\xi_i(\omega)\in B\}\cap\{\tau_i(\omega)<t\}]\in\mathscr{B}_t\end{aligned}$$

for each $B\in\mathscr{B}(R^n)$ by the above lemma.

Next let $v^n=\{\mu_1^n,\eta_1^n;\mu_2^n,\eta_2^n,\ldots\}$ be a sequence of impulsive controls and τ be a stopping time. Put

$$\begin{aligned}\bar{\mu}_i(\omega)&=\tau(\omega)+\mu_i^n(\theta_\tau\omega)\qquad\text{if } X_\tau(\omega)\in U_{x_n}(\delta)\\\bar{\eta}_i(\omega)&=\eta_i^n(\theta_\tau\omega)\qquad\text{if } X_\tau(\omega)\in U_{x_n}(\delta),\end{aligned}$$

where $U_{x_n}(\delta)$ is a neighborhood of x_n introduced in the proof of Lemma 1.2. Then we can prove that $\bar{v}=\{\bar{\mu}_1,\bar{\eta}_1;\bar{\mu}_2,\bar{\eta}_2;\ldots\}$ is also an impulsive control. Since $\bar{\tau}_i^n(\omega)$

$=\tau+\mu_i^n(\theta_\tau\omega)$ is a stopping time and $\mathscr{B}_\tau\subset\mathscr{B}_{\overline{\tau}_i^n}$ for each n and i,

$$\{\bar{\mu}_i(\omega)<t\}=\bigcup_n[\{X_\tau(\omega)\in U_{x_n}(\delta)\}\cap\{\overline{\tau}_i^n<t\}]\in\mathscr{B}_t$$

for each i. Therefore we have only to prove that

$$\{\bar{\eta}_i(\omega)\in B\}\cap\{\bar{\mu}_i(\omega)<t\}$$
$$=\bigcup_n[\{X_\tau\in U_{x_n}(\delta)\}\cap\{\eta_i^n(\theta_\tau\omega)\in B\}\cap\{\tau+\mu_i^n(\theta_\tau\omega)<t\}]\in\mathscr{B}_t$$

for each i and $B\in\mathscr{B}(R^n)$. It holds that

$$\{X_\tau\in U_{x_n}(\delta)\}\cap\{\eta_i^n(\theta_\tau\omega)\in B\}\cap\{\tau+\mu_i^n(\theta_\tau\omega)<t\}$$
$$=\bigcup_{\substack{0\leqq s<t\\ \text{rational}}}[\{X_\tau\in U_{x_n}(\delta)\}\cap\{\eta_i^n(\theta_\tau\omega)\in B\}\cap\{\mu_i^n(\theta_\tau\omega)<s\}\cap\{\tau<t-s\}]$$
$$=\bigcup_{\substack{0\leqq s<t\\ \text{rational}}}[\{X_\tau\in U_{x_n}(\delta)\}\cap\{\tau<t-s\}\cap\theta_\tau^{-1}(\{\eta_i^n\in B\}\cap\{\mu_i^n<s\})].$$

$$\{X_\tau\in U_{x_n}(\delta)\}\cap\{\tau<t-s\}\in\mathscr{B}_{t-s}\subset\mathscr{B}_t$$

and

$$\{\tau+s<t\}\cap\theta_\tau^{-1}[\{\eta_i^n\in B\}\cap\{\mu_i^n<s\}]\in\mathscr{B}_t$$

because

$$\theta_\tau^{-1}[\{\eta_i^n\in B\}\cap\{\mu_i^n<s\}]\in\mathscr{B}_{\tau+s}.$$

Hence we get our conclusion.

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An Extension of Expectation

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1. Introduction

Let $\mathcal{F}$ denote the convolution semigroup of probability distributions (the operation is the convolution, denoted by $*$, corresponding to the addition of independent random variables). The expectation, denoted by $\mathbf{E}(F)$, satisfies

$$\mathbf{E}(F_1 * F_2) = \mathbf{E}(F_1) + \mathbf{E}(F_2)$$

whenever the expectations on the right-hand side are finite, that is, it is a partial homomorphism. Our aim is to derive the following

Theorem. *There exists a homomorphism $\varphi: \mathcal{F} \to R^+$ (R^+ denotes the additive group of real numbers) such that $\varphi(F) = \mathbf{E}(F)$ for every F having a finite expectation.*

The difficulty is caused by the fact that $\mathcal{F}$ is not cancellative, i.e. $F * F_1 = F * F_2$ does not imply $F_1 = F_2$ (the Khinčin phenomenon, see e.g. Feller [2], p. 479). In Sect. 2 we develop an algebraical method to eliminate this problem, in Sect. 3 we apply it to our case and in Sect. 4 we state several further results and open problems.

2. An Algebraical Lemma

Lemma. *Let S be a commutative semigroup, S' a subsemigroup of S, G a group and $\psi: S' \to G$ a homomorphism. For the existence of a homomorphism $\varphi: S \to G$ satisfying $\varphi(s) = \psi(s)$ for every $s \in S'$ the following condition is necessary:*

$$ss_1 = ss_2 \quad \text{implies} \quad \psi(s_1) = \psi(s_2) \quad \text{if } s \in S, s_1, s_2 \in S'. \tag{1}$$

If G is divisible, then (1) is also sufficient.

This may be well-known, but we cannot give any reference.

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Proof. The necessity of (1) is evident. To prove sufficiency write $s_1 \sim s_2$ for $s_1, s_2 \in S$ if there is an $s \in S$ such that $ss_1 = ss_2$. $\sim$ is evidently a congruence relation, and the factor-semigroup $S_0 = S/\sim$ is easily seen to be cancellative. Let $\alpha: S \to S_0$ denote the canonical homomorphism. Our condition (1) just means that ψ is compatible with $\sim$, and hence it induces a homomorphism $\psi_0: S_0' \to G$, where $S_0' = \alpha(S')$, such that $\psi = \psi_0 \circ \alpha$.

S_0, being cancellative, can be embedded into a group H and ψ_0 can be extended to a homomorphism $\psi_1: H' \to G$, where H' is the subgroup of H generated by S_0'. As G is divisible, by a theorem of Baer [1] (see also Fuchs [3], Chap. IV. §21) this homomorphism can be extended to a homomorphism $\varphi_1: H \to G$, and now we can set $\varphi = \varphi_1 \circ \alpha$.

3. Proof of the Theorem

By the Lemma we have to show only that $F_1 * F = F_2 * F$ implies $\mathbf{E}(F_1) = \mathbf{E}(F_2)$ for $F, F_1, F_2 \in \mathcal{F}$. Denote the characteristic functions of F, F_1, F_2 by f, f_1, f_2 respectively. As the expectations are finite, we have

$$\mathbf{E}(F_j) = -if_j'(0) \quad (j = 1, 2). \tag{2}$$

$F * F_1 = F * F_2$ implies $ff_1 = ff_2$. f is continuous and assumes the value 1 at 0, hence we have $f_1(t) = f_2(t)$ for sufficiently small values of t, and therefore

$$f_1'(0) = f_2'(0). \tag{3}$$

(2) and (3) imply $\mathbf{E}(F_1) = \mathbf{E}(F_2)$.

4. Problems and Results

1) It is natural to ask whether such an extension can have any "good" properties. It is easy to see that it cannot be monotonic; however, we proved that it may have the weaker property of being positive for positive variables and negative for negative ones.

2) The additivity of expectation is not restricted to independent variables. Can a real number $\varphi(\xi)$ be assigned to every random variable ξ being additive (i.e. satisfying $\varphi(\xi_1 + \xi_2) = \varphi(\xi_1) + \varphi(\xi_2)$ for all ξ_1 and ξ_2), depending only on the distribution of ξ and coinciding with the expectation if it exists? We conjecture that the answer is negative.

3) Besides its additivity, for independent variables the expectation is also multiplicative. Can an extension have this property as well? That is, together with the additive convolution $*$ we may also consider the multiplicative convolution $\circledast$, defined by $F_1 \circledast F_2 = F$ if there are independent variables having distributions F_1 and F_2, whose product has a distribution F. The operations $*$ and $\circledast$ turn $\mathcal{F}$ into a quasi-ring (not a ring, as $*$ does not have an inverse). Does there exists a mapping $\varphi: \mathcal{F} \to R$ satisfying (i) $\varphi(F_1 * F_2) = \varphi(F_1) + \varphi(F_2)$, (ii) $\varphi(F_1 \circledast F_2) = \varphi(F_1)\varphi(F_2)$, (iii) $\varphi(F) = \mathbf{E}(F)$ whenever the expectation is finite? We

can show that the latter two conditions can be satisfied if (i) is dropped.

4) As the variance (or any other cumulant) is also a partial homomorphism and is completely determined by the behaviour of the characteristic function at the neighbourhood of 0, it can also be extended for every variable. However, if we seek this extension of variance in the form

$$\delta(\xi)=\varphi((\xi-\varphi(\xi))^2) \tag{4}$$

where φ is an extended expectation, we generally fail. Can one construct φ so that the mapping δ defined by (4) will be also a homomorphism? This is connected with problems 2) and 3).

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Probabilistic Behaviour of Some Lacunary Series

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1. Preliminaries

Let $\{n_k\}_1^\infty$ be a sequence of integers such that

$$n_{k+1}/n_k \geqq q > 1 \quad \text{for each } k.$$

For a real sequence $\{a_k\}$ and θ in $[0, 1]$ we consider the series

$$C_n(\theta) = \sum_{k=1}^{n} a_k \cos(2\pi n_k \theta), \tag{1}$$

$$S_n(\theta) \sum_{k=1}^{n} a_k \sin(2\pi n_k \theta), \tag{2}$$

and

$$Z_n(\theta) = \sum_{k=1}^{n} a_k \exp(2\pi i n_k \theta). \tag{3}$$

By Sidon's theorem a necessary and sufficient condition for the everywhere convergence of these series to a bounded function is that $\sum_1^\infty a_k$ converges absolutely. A necessary and sufficient condition for the almost everywhere convergence is that $\sum_1^\infty a_k^2$ converges. The fact that $\{\exp(2\pi i n_k \theta)\}_1^\infty$ behaves somewhat like a sequence of independent identically distributed random variables (see [6] and [7]) has led to a number of probabilistic results on the asymptotic behaviour of these series when $\sum_1^\infty a_k^2$ diverges. The results that have been obtained (e.g. the central limit theorem [10] and the law of the iterated logarithm [11] and [12]) have usually been of more interest to probabilists than to complex analysts.

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The explanation for this probabilistic behaviour is to be found in the following invariance principle for lacunary series obtained by Philipp and Stout (see [9], p. 60). Let

$$\tfrac{1}{2}B_N = A_N^2 = \tfrac{1}{2}\sum_{k\leqq N} a_k^2$$

and if $\omega\in[0,1]$ define

$$S(t,\omega) = \sum_{k\leqq N} a_k \cos(2\pi n_k \omega) \text{ when } t\in[A_N^2, A_{N+1}^2).$$

Proposition 1. *Suppose that $A_N\to\infty$ as $N\to\infty$ and that there exists $\delta\in(0,1]$ such that $a_k = o(1)\,A_k^{1-\delta}$ as $k\to\infty$. Then without changing the distributions of S we can redefine the process $\{S(t), t\geqq 0\}$ on a richer probability space together with a standard brownian motion $X(t)$ such that*

$$|X(t)-S(t)| = o(1)\,t^{\frac{1}{2}-\lambda} \text{ as } t\to\infty \tag{4}$$

for each $\lambda < \delta/32$.

In [9] it is shown how the central limit theorem and the law of the iterated logarithm are immediate consequences of this result.

Our present work was motivated by a desire to find conditions on $\{a_j\}$ which are necessary and sufficient for the almost everywhere divergence to infinity of the series (1), (2) and (3). In [8], p. 157, Kahane showed that if X is a one dimensional brownian motion and $\{t_j\}$ is an increasing sequence satisfying

$$t_{n+1} - t_n \geqq t_n - t_{n-1} \text{ for } n = 1, \ldots,$$

then $\sum t_n^{-\frac{1}{2}} < \infty$ implies that $|X(t_n)|\to\infty$ almost surely, whilst if $\sum t_n^{-\frac{1}{2}} = \infty$ then the sequence $\{X(t_n)\}$ is almost surely dense on the line. Similar behaviour for the complex brownian motion is governed by the condition $\sum t_j^{-1} < \infty$. *Were it not for the error term* in (4), the invariance principle, with $t_n = 2B_n$ and $\{a_j\}$ monotone, could be applied to yield

$\sum B_N^{-\frac{1}{2}} < \infty$ implies $|C_n(\theta)|\to\infty$ for almost all θ whilst $\sum B_N^{-\frac{1}{2}} = \infty$ implies that for almost all θ the sequence $\{C_n(\theta)\}$ is dense on the line.

Similar results could be deduced for S_n and Z_n. Unfortunately, even if the condition $\lambda < \delta/32$ could be replaced by $\lambda < \delta$, the error term is too large to allow us to obtain a result of this type. However, these observations do point the way to the truth of the matter.

In this paper we obtain a direct approximation to $(n_k x) = n_k x - [n_k x]$, by independent random variables each uniformly distributed over $[0,1]$. Our results are applied in Sect. 3 to give information regarding the almost everywhere behaviour of lacunary trigonometric and other lacunary sums. In Sect. 4 we change from considering the almost sure asymptotic behaviour of sums to looking at the θ sets where series have exceptional behaviour. In these cases we consider the Hausdorff dimensions of the exceptional sets. We also give some applications to lacunary power series.

2. Approximation to $(n_k x)$

Let $[x]$ denote the integer part of x, and $(x)=x-[x]$. Let $\{n_k\}_1^\infty$ be a sequence of integers. We say that $\{n_k\}$ is strongly lacunary if $n_{k+1}/n_k>1$ for all k and $\sum_1^\infty n_k/n_{k+1}$ is finite. Thus, for example, $k!$ is not strongly lacunary but $(k!)^2$ is strongly lacunary. Strongly lacunary series were used, implicitly, in [4].

Theorem 1. *Let $\{n_k\}$ be a strongly lacunary sequence of integers. If $0<\delta<1$, there exist:*

a) *a sequence of independent random variables $\{\theta_k\}$, each uniformly distributed over $[0,1]$ and defined on some probability space Ω;*

b) *a subset A_δ of $[0,1]$ of measure less than δ;*

and

c) *an invertible map $T:\Omega\to[0,1]\setminus A_\delta$ such that T and T^{-1} take sets of positive measure into sets of positive measure and such that for $x\notin A_\delta$*

$$|(n_k x)-\theta_k(T^{-1}x)|\leqq 3n_k/n_{k+1}$$

provided $k\geqq p=p(\delta)$.

Proof. Let $m_k=[n_{k+1}/n_k]-1$, and $I(k)$ be the family of intervals $[j/n_k,(j+1)/n_k)$, $j=0,1,\ldots,n_k-1$. Then each interval of $I(k)$ contains m_k (or, (possibly, m_k+1) intervals of $I(k+1)$. We fix a large integer p and define, inductively, a family $J(k)$, $k\geqq p$ of intervals. Let

$$J(p)=I(p).$$

If $J(k)$ has been defined we let $\bar{J}(k+1)$ consist of all those intervals of $I(k+1)$ which are strictly contained in some interval of $J(k)$. Thus $\bar{J}(k+1)$ consists of clusters of m_k (or m_k+1) intervals of length n_{k+1}^{-1}. For each cluster which has m_k+1 members we remove the interval furthest to the right. The resulting family is called $J(k+1)$.

We define $K(p)$ to be the intersection over $k\geqq p$ of the union of all the intervals in $J(k)$. Then $K(p)$ is a generalized Cantor set and the Lebesgue measure, $\lambda(K(p))$, of $K(p)$ satisfies

$$\lambda[K(p)]\geqq 1-2\sum_{k\geqq p} n_k/n_{k+1}$$

so that

$$\lambda[K(p)]\to 1 \quad \text{as} \quad p\to\infty.$$

If $x\in K(p)$ we define

$$\alpha_p(x)=[n_p x]$$

and if $k>p$ we let $\alpha_k(x)$ be the number of consecutive intervals of $I(k+1)$ wholly to the left of x, which are contained in the same interval of $J(k)$ as x. Then

$$\alpha_k(x)\in\{0,1,\ldots,m_k-1\}$$

and

$$|\alpha_k(x)/m_k - (n_k x)| \leqq 2/m_k. \tag{5}$$

Now define $N_p = n_p$ and, for $k > p$, $N_k = m_k N_{k-1}$. Let $Y = [0, 1]$ be a second copy of the unit interval and define

$$\beta_p(y) = [N_p y]$$

and, for $k > p$,

$$\beta_k(y) = [N_k y] \bmod m_k.$$

The map

$$V: K(p) \to Y$$

defined by

$$Vx = y$$

if and only if

$$\alpha_k(x) = \beta_k(y) \quad \text{for } k \geqq p, \tag{6}$$

is such that V and its inverse take sets of positive measure into sets of positive measure. Under Lebesgue measure on Y the family $\{\beta_k\}\, k \geqq p$ are independent. It will now be shown that β_k/m_k can be approximated by independent random variables, uniformly distributed over $[0, 1]$.

Let $\{\phi_k\}_p^\infty$ be independent identically distributed random variables, each uniformly distributed over $[0, 1]$. Let

$$\theta_k = (N_k \phi_k) \tag{7}$$

so that $\{\theta_k\}_p^\infty$ are also independent and uniformly distributed over $[0, 1]$. Let

$$\gamma_k = [N_k \phi_k] \bmod m_k.$$

Then

$$|\gamma_k m_k - \theta_k| \leqq 1/m_k \tag{8}$$

and $\{\gamma_k\}_p^\infty$ have the same joint distributions as $\{\beta_k\}_p^\infty$. Thus if Ω is a probability space supporting $\{\phi_k\}_p^\infty$ there exists a measure theoretic isomorphism between Y and Ω. The result now follows from statements (5) to (8).

3. Oscillation of Series

Suppose that $\{X_i\}_1^\infty$ are independent identically distributed R^d valued random variables. For a fixed sequence $\{a_j\}$ we define the weighted partial sums

$$T_n = \sum_1^n a_i X_i.$$

We shall assume that the sequence (a_j) satisfies the following conditions:

$$a_i \geqq 0 \quad \text{for all } i; \tag{9}$$

$$B_n \to \infty \text{ and } \max_{1 \leqq i \leqq n} a_i = o(1)\, C_n \text{ as } n \to \infty \tag{10}$$

where

$$B_n = \sum_i^n a_i^2 \quad \text{and} \quad C_n = \sum_1^n a_i. \tag{11}$$

We say that the sequence is d-smooth if

$$\sum_{m=1}^{N} \sum_{n=1}^{m} (B_n B_m - B_n^2)^{-\frac{1}{2}d} = O(1) \left(\sum_{n=1}^{N} B_n^{-\frac{1}{2}d} \right)^2 \text{ as } N \to \infty. \tag{12}$$

Note that if $\{a_j\}$ is monotone increasing or if $a_j = j^\alpha, \alpha > -\frac{1}{2}$ then a_j is d-smooth for $d=1$ or $d=2$.

The following proposition can be proved by combining the asymptotic estimates for the distribution of T_n (obtained by Bretagnolle and Dacunha-Castelle in [2]) with Kahane's argument on p. 122–124 of [8].

Proposition 2. *Suppose that a_j is as above, and that $\{X_i\}$ have finite variance and mean zero. If*

$$\sum_{n=1}^{\infty} B_n^{-\frac{1}{2}d} < \infty$$

then

$$|T_n| \to \infty \text{ with probability one.}$$

If $d=1$ or $d=2$, a_j is d-smooth, and

$$\sum_{n=1}^{\infty} B_n^{-\frac{1}{2}d} = \infty$$

then, with probability one,

$$\{T_n\} \text{ is dense in } R^d.$$

Since we can approximate $(n_k \theta)$ by independent random variables, Proposition 2 allows us to deduce results on the asymptotic behaviour of lacunary series. For the rest of this section we shall suppose that $\{n_k\}$ satisfies the conditions

$$\sum_1^{\infty} n_k / n_{k+1} < \infty \tag{13}$$

and

$$\sum_1^{\infty} a_k n_k / n_{k+1} < \infty. \tag{14}$$

Theorem 2. *Suppose that* $\{a_k\}$ *satisfies the conditions* (9) *to* (11) *and* (12) *for* $d=1$, *and that* $\{n_k\}$ *satisfies* (13) *and* (14). *Let* $S_n(\theta)$ *and* $C_n(\theta)$ *be as defined in* (2) *and* (1). *Then if*

$$\sum_1^\infty B_n^{-\frac{1}{2}} < \infty$$

$$|S_n(\theta)| \to \infty \text{ and } |C_n(\theta)| \to \infty \text{ as } n \to \infty$$

for almost all θ. *If*

$$\sum_1^\infty B_n^{-\frac{1}{2}} = \infty$$

then, for almost all θ, *the sequences* $\{S_n(\theta)\}$ *and* $\{C_n(\theta)\}$ *are dense on the line.*

Proof. Consider the sequence $\{S_n(\theta)\}$. Suppose that $0<\delta<1$ and that A_δ, T, Ω and $\{\theta_i\}$ are as in Theorem 1. Let

$$X_i(\omega) = \sin(2\pi\theta_i(\omega)) \text{ on } \Omega$$

and

$$Y_i(x) = \sin(2\pi\theta_i(T^{-1}x)) \text{ on } [0,1]\backslash A_\delta.$$

The properties of T ensure that

$$\sum_1^n a_i X_i$$

and

$$\sum_1^n a_i Y_i$$

have the same almost sure asymptotic behaviour on their respective domains.

Now

$$|\sin(2\pi n_k x) - \sin(2\pi\theta_k(T^{-1}x))| \leqq 2\pi|(n_k x) - \theta_k(T^{-1}x)|$$
$$\leqq 2\pi n_k/n_{k+1}$$

provided k is sufficiently large and that $x \notin A_\delta$. Thus, by (14), if $x \notin A_\delta$

$$S_n(x) - \sum_1^n a_i Y_i(x)$$

converges to a finite limit as $n \to \infty$. Thus $S_n(x)$ has the same almost sure asymptotic behaviour as $\sum_1^n a_i Y_i(x)$, and so has the same almost sure asymptotic behaviour as $\sum_1^n a_i X_i$. The result, for $S_n(\theta)$, now follows from the facts that δ is arbitrary and that we can apply Proposition 2 to the random sum. A similar argument gives the result for $C_n(\theta)$.

Note that the same argument applies if we replace $\sin x$ and $\cos x$ by any non trivial periodic function which satisfies a Lipschitz condition of order one. The argument can also be used to give the following result on the behaviour of complex sums.

Theorem 3. *Suppose that* $\{a_k\}$ *satisfies the conditions* (9) *to* (11) *and* (12) *for* $d=2$, *and that* $\{n_k\}$ *satisfies* (13) *and* (14). *Let* $Z_n(\theta)$ *be as defined in* (3). *Then if*

$$\sum_1^\infty B_n^{-1} < \infty$$

$$|Z_n(\theta)| \to \infty \quad \text{as } n \to \infty$$

for almost all θ. *If*

$$\sum_1^\infty B_n^{-1} = \infty$$

then, for almost all θ, *the sequence* $\{Z_n(\theta)\}$ *is dense in the plane.*

It would be interesting to know to what extent these results are valid for general lacunary series. Presumably the condition on $\{n_k\}$ can be relaxed somewhat but the fact that the finiteness of $\sum B_n^{-1}$ may change if we reorder the coefficients shows that some restriction on $\{a_j\}$ is necessary. It seems difficult to get very far by using complex variable methods. The problem lies in the fact that we have no asymptotic estimates for

$$P\{Z_n \in D(a,b)\}$$

($D(a,b)$ being the disc centre a with radius b). The central limit theorem only gives estimates for

$$P\{Z_n B_n^{-1} \in D(a,b)\}.$$

Even the following problem does not seem to have been solved.
Let

$$Q_n(\theta) = \sum_1^n \exp(2\pi i n_k \theta).$$

Is it true that whenever $\{n_k\}$ is lacunary $\{Q_n(\theta)\}$ is, for almost all θ, dense in the plane? When $n_k=(k!)^2$ the answer is yes (by Theorem 3). Complex variable methods can be used to show that there exists $r>0$ such that for almost all θ

$$\limsup_{n\to\infty} D(Q_n(\theta), r) = R^2.$$

4. Asymptotic Behaviour

In this section we consider the exceptional behaviour of the series

$$Z_N(\theta) = \sum_1^N \exp(2\pi i n_k \theta). \tag{15}$$

The results we obtain are special cases of those in the next section but the simpler proof serves to motivate later arguments.

The Hausdorff measure μ^α is defined by $\mu^\alpha = \lim_{\delta \to 0+} \mu^\alpha_\delta$ where

$$\mu^\alpha_\delta(A) = \inf \sum |S_i|^\alpha$$

and the infimum is taken over all countable covers of A by sets, S_i, of diameter $|S_i|$ less than δ. The Hausdorff dimension of a set B is then given by

$$\dim B = \inf\{\alpha : \mu^\alpha(B) = 0\}.$$

Suppose that $\lambda \in (0,1)$ and define

$$S(\lambda) = \bigcup_{N=1}^{\infty} \{\theta : \cos(2\pi n_k \theta) \geqq \lambda \text{ for all } k \geqq N\}.$$

Standard arguments show that, if $\inf n_{k+1}/n_k > 3$, $S(\lambda)$ has positive dimension for some λ, and that, if $n_{k+1}/n_k \to \infty$, $S(\lambda)$ has dimension one for each λ.

It follows from (15) that for all θ

$$|Z_N(\theta)| \leqq N$$

and the law of the iterated logarithm for exponential sums (see [11]) ensures that for almost all θ

$$Z_N(\theta) = O(1)(N \log\log N)^{\frac{1}{2}} \quad \text{as } N \to \infty.$$

On the other hand Zygmund, in [12], has shown that

$$\limsup_{N \to \infty} |Z_N(\theta)|/N > 0$$

on a set that is dense in $[0,1]$. In [5] Erdös and Taylor showed that this exceptional behaviour takes place on a set of dimension one. The fact that

$$\liminf_{N \to \infty} |Z_N(\theta)|/N \geqq \lambda \quad \text{on } S(\lambda)$$

shows that an even greater growth rate is possible when $\inf n_{k+1}/n_k > 3$.

We now consider the special case $n_k = 2^k$. By looking at the points $k2^{-p}$, $k = 1, 2, \ldots, 2^p$, $p = 1, 2$, we see that

$$Z_N(\theta) = N + O(1)$$

on a countable dense set. More generally we have:

Theorem 4. *Suppose that $Z_N(\theta)$ is given by* (15) *and that $n_k = 2^k$. Then*

$$\{\theta : Z_N(\theta) \sim cN \text{ as } N \to \infty, \text{ for some } c \neq 0\}$$

has Hausdorff dimension one.

Proof. Suppose that $0<p<1$ and that $q=1-p$. Let $\{X_i\}_1^\infty$ be a sequence of independent random variables each with distribution given by

$$P(X_i=0)=q \quad \text{and} \quad P(X_i=1)=p.$$

Let

$$X=\sum_{i=1}^{\infty} X_i 2^{-i}$$

and let m be the measure on $[0,1]$ defined by

$$m(A)=P(X\in A).$$

Since

$$q\int_0^1 \sin(2\pi x)\,dm(x)=(q-p)\int_0^{\frac{1}{2}} \sin(2\pi x)\,dm(x)$$

we have

$$\int_0^1 \exp(2\pi i x)\,dm(x)=0$$

if and only if $p=q$.

Let

$$B=\left\{x: \sum_1^N [2^k x]/N \to p \text{ as } N\to\infty\right\}.$$

Billingsley (see [1] p. 141 onwards) shows that if A is such that $m(B\cap A)>0$ then

$$\dim(B\cap A)=e(p)=-(p\log p+q\log q).$$

Next consider the transformation $Tx=2x \bmod 1$. The following facts can easily be checked:

i) m is supported by B (use the strong law of large numbers);

ii) $TB\subset B$; and

iii) T preserves m and is ergodic with respect to m (as is shown by a modification of the argument on p. 12 of [1]). Thus the ergodic theorem ensures that

$$\sum_1^N \exp(2\pi i T^k x)/N \to \int_0^1 \exp(2\pi i x)\,dm(x)$$

for m almost all x. Thus if $p\neq q$

$$\sum_1^N \exp(2\pi i 2^k x)/N \to c\neq 0$$

on a set of dimension at least $e(p)$. As $\sup\{e(p)\}=1$ the result follows.

5. Asymptotic Behaviour (General Case)

Now let $\{a_k\}$ be given a sequence with $a_k\geqq 0$ and set

$$Z_N(\theta)=\sum_1^N a_k \exp(2\pi i 2^k \theta).$$

We define C_N by

$$C_N = \sum_1^N a_j.$$

Theorem 5. *Suppose that* $C_N \to \infty$ *as* $N \to \infty$. *Then if, as* $N \to \infty$

$$a_N = O(1)\, C_N^{1-\delta} \qquad \text{for some } \delta > 0,$$

the set

$$\{\theta\colon Z_N(\theta) \sim \lambda\, C_N \text{ as } N \to \infty \text{ for some } \lambda \neq 0\}$$

has Hausdorff dimension one.

Proof. Let X be the random variable constructed in Theorem 4 and let $m_n(X, dx)$ be the random measure obtained by putting atoms of mass a_j/C_n at the points $2^j X \bmod 1$, $j = 1, 2, \ldots, n$. First we show that

$$P\{m_n \text{ converges weakly to } m\} = 1. \tag{16}$$

It is enough to check that for each dyadic cylinder $I = [k2^{-p}, (k+1)2^{-p})$ we have

$$m_n(I) \to m(I) \qquad \text{almost surely.}$$

Then the fact that there are only a countable number of such cylinders means that this convergence takes place for all I with probability one, which implies (16).

Define $\varepsilon_n(x) = [2^n x] \bmod 2$. Then any dyadic cylinder I can be specified by a sequence $\{\lambda_k\}_1^p$, $\lambda_k = 0$ or 1, by means of

$$I = \{x\colon \varepsilon_k(x) = \lambda_k;\ k = 1, 2, \ldots, p\}.$$

Define the random variable Y_i to be the indicator of the event

$$X_{i+k} = \lambda_k \qquad k = 1, 2, \ldots, p.$$

Then $EY_i = m(I)$ and Y_i and Y_j are independent if $|i-j| > p$.

If we apply Theorem 1.1.15 of Theodorescu [5] to the random variables $f_i = a_i Y_i$ we see that

$$m_n(I) = \sum_{i=1}^n a_i Y_i / C_n$$

$$\to m(I)$$

almost surely, provided that $\sum (a_j/C_j)^2$ converges. However for some K and some $\delta > 0$ we have $a_j \leqq K C_j^{1-\delta}$ for all j. Thus

$$\sum (a_j/C_j)^2 \leqq K \sum a_j\, C_j^{-(1+\delta)}$$

which is finite by Dini's theorem.

Thus, for m almost all X, $m_n(X, dy) \to m(dy)$ weakly. Hence

$$\int_0^1 \exp(2\pi i x)\, m_n(X, dx) \to \int_0^1 \exp(2\pi i x)\, dm(x)$$

for m almost all X. If $p \neq q$ the latter limit is non zero and the conclusion follows in the same way as it did in Theorem 4.

6. Lacunary Power Series

We now obtain some results for a class of lacunary power series. First we let

$$S(r,\theta)=\sum_1^\infty a_k r^{n_k}\exp(2\pi i n_k \theta)$$

and

$$M(r)=\sup_\theta S(r,\theta)=\sum_1^\infty a_k r^{n_k}.$$

Lemma. *Suppose that* $a_n \geqq 0$, *that*

$$a_{n+1}\sim a_n \quad \text{as } n\to\infty$$

and that

$$\max_{1\leqq i\leqq n} a_i = o(1)\, C_n \quad \text{as } n\to\infty.$$

Let $N(r)=\max\{k: r^{n_k}\geqq \tfrac{1}{2}\}$. *Then, for all* θ,

$$|Z_{N(r)}(\theta)-S(r,\theta)|=o(1)\, C_{N(r)} \quad \text{as } r\to 1-.$$

Proof. First choose an increasing sequence $\{p_n\}$ of integers such that

$$p_n\to\infty \quad \text{and} \quad a_n p_n = o(1)\, C_n \quad \text{as } n\to\infty,$$

and

$$\tfrac{1}{2}\leqq a_{n+i}/a_n \leqq 2 \text{ provided } |i|\leqq p_n.$$

We define $p(r)$ by

$$p(r)=p_{N(r)},$$

and, where there is no confusion, we write p for $p(r)$ and N for $N(r)$. Now

$$1-r^{n_j}\leqq n_j(1-r).$$

The fact that $r^{N(r)}\geqq \tfrac{1}{2}$ shows that

$$r\geqq \exp(-\log 2/N(r))$$

and so

$$1-r\leqq \log 2/N(r).$$

Thus

$$1-r^{n_j}\leqq (n_j/n_{N(r)})\log 2.$$

Now we express $S(r,\theta)$ in the form

$$S(r,\theta)=\sigma_1(r,\theta)+\sigma_2(r,\theta)+\sigma_3(r,\theta) \tag{17}$$

where

$$\sigma_1(r,\theta)=\sum_{j\leqq N-p} a_j r^{n_j}\exp(2\pi i n_j\theta),$$

$$\sigma_2(r,\theta)=\sum_{|j-N|<p} a_j r^{n_j}\exp(2\pi i n_j\theta),$$

and

$$\sigma_3(r,\theta)=\sum_{j\geqq N+p} a_j r^{n_j}\exp(2\pi i n_j\theta).$$

Comparison with a geometric series shows that

$$\begin{aligned}|\sigma_3(r,\theta)| &= O(1)\,a_{N+p}\\ &= O(1)\,a_N\\ &= o(1)\,C_{N(r)}.\end{aligned}$$

Also

$$\begin{aligned}|\sigma_2(r,\theta)| &\leqq 2p(r)\max_{|j-N|<p} a_j\\ &\leqq 4p(r)\,a_{N(r)}\\ &= o(1)\,C_{N(r)}.\end{aligned}\tag{18}$$

Now

$$\begin{aligned}|\sigma(r,\theta)-Z_{N-p}(\theta)| &\leqq \sum_{j\leqq N-p}(1-r^{n_j})\,a_j\\ &\leqq \sum_{j\leqq N-p}(n_j/n_N)\,a_j\\ &\leqq (n_{N-p}/n_N)\sum_{j\leqq N-p} a_j\\ &\leqq q^{-p}\,C_N\\ &= o(1)\,C_{N(r)} \quad \text{as } r\to 1-.\end{aligned}\tag{19}$$

Finally

$$\begin{aligned}|Z_N(\theta)-Z_{N-p}(\theta)| &\leqq \sum_{N-p}^{N} a_j\\ &\leqq 2(p(r)+1)\,a_{N(r)}\\ &= o(1)\,C_{N(r)} \quad \text{as } r\to 1-.\end{aligned}\tag{20}$$

The result follows by combining (17), (18), (19) and (20).

Let $B(r)=(\sum a_k^2 r^{2n_k})^{\frac{1}{2}}$. Then, according to Weiss [12], for almost all θ we have

$$\limsup_{r\to 1-}|S(r,\theta)|\,[B(r)^2\log\log B(r)]^{-\frac{1}{2}}=1.$$

We can now examine the exceptional set for a special class of lacunary series.

Theorem 6. *Suppose that $n_k=2^k$, that $a_k\geqq 0$, and that $a_k\sim a_{k+1}$ as $k\to\infty$. Then if, for some $\delta>0$,*

$$a_k=O(1)\,C_k^{1-\delta} \quad \text{as } k\to\infty$$

the set

$$\{\theta\colon S(r,\theta)\sim\lambda M(r) \quad \text{as } r\to 1- \text{ for some } \lambda\neq 0\}$$

has Hausdorff dimension one.

Proof. The preceding lemma with $\theta=0$ shows that

$$M(r)\sim C_{N(r)} \quad \text{as } r\to 1-.$$

The result then follows from Theorem 5.

This result admits interpretations in the language of cluster sets. For example let $f(z)=\sum_0^\infty z^{2^n}$. The unit circle is then the natural boundary of f and it can be shown that almost all points of the unit circle are Plessner points (see [3], p. 147). Our result shows that the set of Fatou points (with limit value ∞) has Hausdorff dimension one.

Remarks. 1. I would like to thank the referee for his valuable comments and suggestions for improvement.

2. The proofs of the results in Sect. 3–6 remain valid if we assume only that $n_k=\prod_{j\leqq k} r_j$ where the r_j are integers which are at least two.

3. We have chosen in this paper to emphasize those problems where probabilistic techniques yield better results than are available through the techniques of complex analysis. Presumably Theorems 3, 5 and 6 hold in much greater generality but at present they seem beyond the range of both probabilistic and complex analytical methods.

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Scaling Transformations for {0, 1}-valued Sequences

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Summary. Let $r \geqq 2$ be an integer and let φ: $\{0,1\}^r \to \{0,1\}$ be a function. Let T be the transformation on $\Omega = \{0,1\}^Z$ given by $(T\omega)(i) = \varphi(\omega(ri), \omega(ri+1), \ldots, \omega(ri+r-1))$ for all $i \in Z$. For P in the class of strongly-mixing shift-invariant measures on Ω, we investigate when P is invariant with respect to T and when $T^n P$ converges. For example if r is odd and $\varphi(\omega_0, \ldots, \omega_{r-1}) = 1$ iff $\sum \omega_i > \frac{1}{2}r$, the invariant measures are the Bernoulli measures with means 0, $\frac{1}{2}$ or 1 and $T^n P$ must converge to one of these three measures. Other choices of φ can give more complicated behaviour.

1. Introduction

Let Z denote the set of integers and let $\Omega = \{0,1\}^Z$. Let $\mathfrak{F}$ denote the σ-algebra of subsets of Ω which is generated by the cylinder sets. Let S: $\Omega \to \Omega$ be the shift transformation on Ω; that is $(S\omega)(i) = \omega(i+1)$. Throughout the following, measures on Ω will be taken to be shift-invariant probability measures on the measurable space $(\Omega, \mathfrak{F})$. We will consider $\omega(i)$ to denote the i'th component of an element $\omega \in \Omega$ or, when convenient, to denote the i'th term of a strictly stationary sequence $\{\omega(i)\}$ of $\{0,1\}$-valued random variables defined on some other probability space. In the latter case, one measure on $(\Omega, \mathfrak{F})$ is the distribution of $\{\omega(i)\}$. The reader should have no difficulty distinguishing between these two interpretations.

Let φ: $\{0,1\}^r \to \{0,1\}$ be a function, where r is an integer greater than one. The *scaling transformation* corresponding to φ is the transformation T: $\Omega \to \Omega$ defined by

$$(T\omega)(i) = \varphi(\omega(ri), \omega(ri+1), \ldots, \omega(ri+r-1)). \tag{1}$$

We call φ the *scaling function* for T and r the *scale* of T. Let P be a measure on Ω and let P_n be the measure on Ω given by

$$P_n(A) = T^n P(A) = P(T^{-n}(A)). \tag{2}$$

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The purpose of this paper is to determine which measures are invariant under T (i.e., to solve the equation $P_1 = P$) and to determine for which P the sequence $\{P_n\}$ converges. The main results of the paper are given in Sect. 3, where we show that for P in a large class, and for certain scaling transformations T, P is T-invariant iff P is one of a finite set of Bernoulli measures. The measures in the set are found by solving a polynomial equation obtained from φ. Furthermore, for such T, P_n converges for all P in the indicated class.

Of particular interest are the *majority transformations* obtained by taking r odd and using the scaling function

$$\varphi(\omega_0, \omega_1, \ldots, \omega_{r-1}) = \begin{cases} 1 & \text{if } \sum_{i=0}^{r-1} \omega_i > \frac{1}{2}r \\ 0 & \text{if } \sum_{i=0}^{r-1} \omega_i < \frac{1}{2}r \end{cases} \tag{3}$$

for all $(\omega_0, \ldots, \omega_{r-1}) \in \{0, 1\}^r$.

The question of identifying the invariant measures for majority transformations was posed by Professor F. Spitzer, who conjectured that the only non-trivial invariant measure is the Bernoulli measure with probability of each outcome being one half. In Sect. 4, we show that these transformations satisfy the requirements of our theorems and that Spitzer was essentially correct. In Sect. 6, we present several other invariant measures which are not in the large class to which our theorems apply.

Majority and other scaling transformations can be used to model a number of physical phenomena. Spitzer, motivated by the work of Mandelbrot (1977) on scaling, considered a person in an airplane observing the earth's surface below as being land or water. Incomplete resolution causes him only to detect the effect of successive intervals of length r. He records a one or zero (land or water) for each interval in a way that depends on the pattern of zeros and ones within the interval. The most natural situation would have him record the value that predominates (i.e. holds the majority) within the interval.

A similar example is obtained by considering a forester who walks through his domain and classifies each section as disease-ridden if the fraction of diseased trees he examines in the section exceeds some specified value.

A third example is obtained by considering a voting system in which the electorate is divided into cells of size r. Decisions are made by first voting within each cell and then having each cell (or its representative) vote in accordance with the majority within the cell. This could be extended to several stages by having cells, subcells, etc. (In practice, of course, representatives do not always heed the majority of their "cells".)

Finally, we note that Wilson (1979) has used a two-dimensional majority transformation to study the Ising model of magnetization.

Another interesting class of scaling transformations are the *sum modulo two transformations* obtained by defining

$$\varphi(\omega_0, \omega_1, \ldots, \omega_{r-1}) \equiv \sum_{i=0}^{r-1} \omega_i \quad \text{(modulo two)}. \tag{4}$$

If we denote T by T_r, we observe a semi-group structure for these transformations:

$$T_r \cdot T_s = T_{rs}.$$

(This equation does not hold in the case of majority transformations.) The sum modulo two transformations are analoguous to the transformations considered by Sinai (1976) and more recently by Taqqu (1979) and others, in connection with self-similar probability measures on $(-\infty, \infty)^Z$. It is easily seen that the hypotheses of the theorems of Sect. 3 do not hold for sum modulo two transformations. The conclusions of the theorems hold for P in a more restricted class but not for all P in the class considered in Sect. 3. These results are given in Sect. 7.

It should be noted that our methods and conclusions are quite different from those of "block transformations" given by

$$(U\omega)(i) = \varphi(\omega(i-\tfrac{1}{2}(r-1)), \omega(i-\tfrac{1}{2}(r-3)), \ldots, \omega(i+\tfrac{1}{2}(r-1)))$$

for odd r. If φ is the majority function, the invariant measures include all measures which are concentrated on sequences all of whose runs have length greater than $\frac{1}{2}r$, since such sequences are fixed points of U. Block transformations have been studied by Coven, Hedlund and Rhodes (1979) and by Miyamoto (1979), among others.

It is obvious that scaling transformations are not invertible. Dekking (1979) investigates transformations which are right inverses of scaling transformations. To be precise, his transformations substitute specified words for each symbol.

2. Classes of Measures

In this section, we give descriptions of some classes of measures on $(\Omega, \mathfrak{F})$ which will play a role in the study of scaling transformations. Recall that measures are assumed to be shift-invariant and to have total mass 1.

The most important measures are the Bernoulli measures. If P is Bernoulli and $P(\omega(0)=1)=p$, we write $P=B_p$. Of course, B_p is the distribution of an i.i.d. sequence $\{\omega(i)\}$ of $\{0,1\}$-valued random variables with $P(\omega(i)=1)=p$.

Let $\mathfrak{F}(j,k)$ denote the σ-algebra of sets in $\mathfrak{F}$ generated by sets of the form

$$\{\omega\in\Omega: \omega(i)=\omega_i, j\leqq i\leqq k\}.$$

We consider the class $\mathcal{M}$ of *strongly mixing* measures on Ω; $P\in\mathcal{M}$ if there exists a function g on the positive integers such that $g(k)\downarrow 0$ as $k\to\infty$ and

$$|P(AB)-P(A)P(B)|\leqq g(k) \tag{5}$$

for all $A\in\mathfrak{F}(-\infty,0)$ and $B\in\mathfrak{F}(k,\infty)$. The Bernoulli measures are the measures in $\mathcal{M}$ for which (5) holds with $g(1)=0$. A second useful subclass of $\mathcal{M}$ is the class $\mathcal{M}_2$ of 2-dependent measures; that is, the set of $P\in\mathcal{M}$ for which (5) holds with $g(2)=0$.

We will say P is a *Markov-based* measure on Ω if P is the distribution of a sequence $\{f(X_n)\}$ where X_n is a stationary aperiodic positive recurrent irreducible Markov chain with a countable state space I and f is a function from I into $\{0,1\}$. It is well known that every Markov-based measure is in $\mathscr{M}$. A special case is obtained by assuming $I=\{0,1\}$ and P is the distribution of $\{X_n\}$ itself.

Various classes of measures on $(\Omega,\mathfrak{F})$ are invariant under scaling transformations. Included are the classes of Bernoulli measures, shift-invariant measures, strongly mixing measures, 2-dependent measures and Markov-based measures. In the last case, the state space of the underlying Markov chain may have to be enlarged if it is finite. The property of being the distribution of an irreducible aperiodic stationary Markov chain on $\{0,1\}$ is not generally preserved. For example, if T is the majority transformation with scale 3 and P is such a distribution and has a symmetric transition matrix, then TP is such a distribution if and only if $P=B_{\frac{1}{2}}$.

3. Invariant Measures and Limits

Lemma 1. *Let $P\in\mathscr{M}$ and let T be a scaling transformation. Every subsequence $\{P_{n_k}\}$ of $\{P_n\}$ has at least one accumulation point (i.e., probability measure to which some subsequence of $\{P_{n_k}\}$ converges weakly). If Q is an accumulation point of $\{P_n\}$, then $Q\in\mathscr{M}_2$.*

Proof. First note that $\mathfrak{F}$ is the Borel σ-algebra with respect to the compact product topology on Ω. By Prohorov's theorem, every subsequence of $\{P_n\}$ has a convergent subsequence (cf. Billingsley (1968), p. 19). We note in passing that $P_{n_k}\Rightarrow Q$ (weak convergence) iff $P_{n_k}(A)\to Q(A)$ for every cylinder set A (Billingsley, p. 37). We make frequent use of these facts without explicit mention.

Let Q be an accumulation point of $\{P_n\}$. Since each P_n is shift-invariant, so is Q. Let $A\in\mathfrak{F}(-\infty,0)$, $B\in\mathfrak{F}(2,\infty)$. Then $T^{-n}A\in\mathfrak{F}(-\infty,r^n-1)$, $T^{-n}B\in\mathfrak{F}(2r^n,\infty)$. Therefore

$$\begin{aligned}|Q(AB)-Q(A)Q(B)| &\leqq \limsup_{n\to\infty}|P_n(AB)-P_n(A)P_n(B)|\\ &=\limsup_{n\to\infty}|P((T^{-n}A)(T^{-n}B))-P(T^{-n}A)P(T^{-n}B)|\\ &\leqq \lim_{n\to\infty} g(r^n)=0.\end{aligned}$$

Thus $Q\in\mathscr{M}_2$. This proves Lemma 1.

It would be nice if we could improve Lemma 1 by showing that each accumulation point Q of $\{P_n\}$ must in fact be a Bernoulli measure. It will be seen in Sect. 7 that this is not always the case. The next result shows that it is the case for some scaling transformations. For each scaling function φ with scale r, we define two sets:

$$\begin{aligned}C=C_\varphi=\{\omega\in\Omega:\ &\varphi(0,\omega(1),\omega(2),\ldots,\omega(r-1))\\ &\neq\varphi(1,\omega(1),\omega(2),\ldots,\omega(r-1))\},\end{aligned}\tag{6}$$

$$C' = C'_{\varphi} = \{\omega \in \Omega : \varphi(\omega(0), \omega(1), \ldots, \omega(r-2), 0) \neq \varphi(\omega(0), \omega(1), \ldots, \omega(r-2), 1)\} \tag{7}$$

Lemma 2. *Let $P \in \mathcal{M}$ and let Q be an accumulation point of the sequence $\{P_n\}$. If*

$$\liminf_{n \to \infty} P_n(C \cap C') < 1, \tag{8}$$

then Q is a Bernoulli measure. This holds in particular if

$$Q(C \cap C') < 1. \tag{9}$$

Proof. We deduce from (8) that either

$$\alpha \equiv \liminf_{n \to \infty} P_n(C) < 1 \tag{10}$$

or the same holds with C replaced by C'. We assume (10), the other case being similar. Let $A \in \mathfrak{F}(-\infty, -1)$ and $B \in \mathfrak{F}(0, 0)$. It suffices to show

$$Q(AB) = Q(A)Q(B). \tag{11}$$

Let $\varepsilon > 0$. Let $\{P_{m_k}\}$ be a subsequence of $\{P_n\}$ such that

$$P_{m_k}(C) < \tfrac{1}{2}(1+\alpha) \tag{12}$$

for all k and such that

$$m_{k+1} - m_k \geqq 2 \tag{13}$$

for all k. Let l be sufficiently large that

$$[\tfrac{1}{2}(1+\alpha)]^l < \varepsilon. \tag{14}$$

Let $\{P_{n_k}\}$ be a subsequence of $\{P_n\}$ which converges to Q and assume

$$n_k > m_k \tag{15}$$

for all k. For k sufficiently large,

$$l g(r^{m_{k-l}}) < \varepsilon. \tag{16}$$

Let $A' = T^{m_{k-l} - n_k}(A)$ and $B' = T^{m_{k-l} - n_k}(B)$. For $\omega \in \Omega$, define $\omega^* \in \Omega$ by

$$\omega^*(i) = \begin{cases} \omega(i) & \text{for } i \neq 0 \\ 1 - \omega(0) & \text{for } i = 0. \end{cases}$$

Then let $B^* = \{\omega \in B' : \omega^* \in B'\} \subseteq B'$. If $\omega \in B' - B^*$, then $T^{n_k - m_{k-l}} \omega \in B$ but $T^{n_k - m_{k-l}} \omega^* \notin B$. This in turn implies that $\omega \in C$, $T\omega \in C, \ldots, T^{n_k - m_{k-l} - 1} \omega \in C$. Since $B \in \mathfrak{F}(0, 0)$, we use (12)–(16) to obtain

$$\begin{aligned} P_{m_{k-l}}(B' - B^*) &\leqq P_{m_{k-l}}(C \cap T^{-1} C \cap \ldots \cap T^{-n_k + m_{k-l} + 1} C) \\ &\leqq P_{m_{k-l}}(C \cap T^{m_{k-l} - m_{k-l+1}} C \cap \ldots \cap T^{m_{k-l} - m_k} C) \\ &\leqq P_{m_{k-l}}(C) P_{m_{k-l+1}}(C) \ldots P_{m_k}(C) + l g(r^{m_{k-l}}) \\ &< 2\varepsilon. \end{aligned}$$

Observe that $B^* \in \mathfrak{F}(1, \infty)$ and $A' \in \mathfrak{F}(-\infty, -1)$. Thus

$$\begin{aligned}|P_{n_k}(AB) - P_{n_k}(A)P_{n_k}(B)| &= |P_{m_{k-1}}(A'B') - P_{m_{k-1}}(A')P_{m_{k-1}}(B')| \\ &\leqq |P_{m_{k-1}}(A'B') - P_{m_{k-1}}(A'B^*)| \\ &\quad + |P_{m_{k-1}}(A'B^*) - P_{m_{k-1}}(A')P_{m_{k-1}}(B^*)| \\ &\quad + P_{m_{k-1}}(A')|P_{m_{k-1}}(B^*) - P_{m_{k-1}}(B')| \\ &\leqq 2P_{m_{k-1}}(B' - B^*) + g(r^{m_{k-1}}) \\ &< 5\varepsilon.\end{aligned}$$

Since ε is arbitrary, (11) follows and Q is Bernoulli. It is obvious that (9) implies (8) so the proof of Lemma 2 is complete.

The requirement that (8) holds is very inconvenient. For some scaling transformations, including the majority transformations, it holds for all $P \in \mathscr{M}$. By contrast, for any scaling transformation whose scaling function φ has the form

$$\varphi(\omega_0, \omega_1, \ldots, \omega_{r-1}) = \omega_0 + \psi(\omega_1, \omega_2, \ldots, \omega_{r-2}) + \omega_{r-1}, \tag{17}$$

where addition is performed modulo 2, the condition fails for all $P \in \mathscr{M}$. The main examples are the sum modulo two transformations. Some partial results for such transformations are given in Sect. 7.

Associated with each scaling function φ is a function $\pi = \pi_\varphi$: $[0,1] \to [0,1]$ defined as follows:

$$\pi(p) = \sum_{i=0}^{r} a_i p^i (1-p)^{r-i} \tag{18}$$

where a_i is the number of r-tuples $(\omega_0, \omega_1, \ldots, \omega_{r-1})$ in $\{0,1\}^r$ which are in $\varphi^{-1}(1)$ and which have exactly i ones and $r-i$ zeros. The fact that $0 \leqq \pi(p) \leqq 1$ for all p follows from the fact that

$$a_i \in \left\{0, 1, \ldots, \binom{r}{i}\right\}$$

for each i, so that in particular

$$\pi(p) \leqq \sum_{i=0}^{r} \binom{r}{i} p^i (1-p)^{r-i} = 1.$$

The importance of the function π is derived from the following fact:

$$TB_p = B_{\pi(p)}. \tag{19}$$

It follows that the Bernoulli measures which are invariant under T are the Bernoulli measures B_p such that p is a fixed point of π, i.e., such that $\pi(p) = p$. (At least one fixed point exists.) Let $\pi^1 = \pi$ and, for $n = 2, 3, \ldots$, let $\pi^n(p) = \pi(\pi^{n-1}(p))$. If $P = B_p$, then $P_n = B_{\pi^n(p)}$. Thus, any limit theorems for $\{P_n\}$ inevitably involve the limiting behaviour of the sequence $\{\pi^n(p)\}$.

Theorem 1. *Let T be a scaling transformation and let $P \in \mathcal{M}$. Assume*

$$P_n(C \cap C') < 1 \tag{20}$$

for some n. For $k = 1, 2, \ldots, P_k = P$ iff $P = B_p$ where $\pi^k(p) = p$. In particular, P is invariant under T iff $P = B_p$ where p is a fixed point of π.

Proof. Suppose $P_k = P$. Then the sequence $\{P_n\}$ is periodic so that (20) implies (8). Also, the subsequence $\{P_{nk}\}$ converges to P as $n \to \infty$. By Lemma 2, $P = B_p$ for some p. The results now follow from (19).

Theorem 2. *Let $P \in \mathcal{M}$ and let T be a scaling transformation. Assume P satisfies* (8). *Also assume π is not the identity function $(\pi(p) \not\equiv p)$ and*

$$\pi^2(p) = p \Leftrightarrow \pi(p) = p \tag{21}$$

for all $p \in [0, 1]$. Then P_n converges to a Bernoulli measure B_p such that $\pi(p) = p$.

Proof. By Lemma 2, there is a subsequence $\{P_{n_k}\}$ of $\{P_n\}$ such that $P_{n_k} \Rightarrow B_p$ for some $p \in [0, 1]$. It follows easily that $P_{n_k + l} \Rightarrow B_{\pi^l(p)}$ for $l = 1, 2, \ldots$. Now Sharovsky (1966) (see also Kloeden, Deakin and Tirkel (1976)) showed that (21) implies $\pi^l(p)$ must converge to a fixed point p_0 of π as $l \to \infty$. Since the set of accumulation points of a sequence is closed, there must be a subsequence $\{P_{m_k}\}$ of $\{P_n\}$ which converges to B_{p_0}. Let $a(n) = P_n(\omega(0) = 1)$. Obviously, $a(m_k) \to p_0$.

The result is trivial if π is a constant function; if π is a constant it must be 0 or 1, which means φ is also constant. Since $\pi(0)$ and $\pi(1)$ can both only equal 0 or 1, there are only two first degree possibilities for π, namely $\pi(p) \equiv p$ and $\pi(p) \equiv 1 - p$, and these are excluded by explicit assumption and (21), respectively. We may therefore assume π has degree two or higher. It follows that there is a neighbourhood N of p_0 such that for all $p \in N - \{p_0\}$,

$$\pi(p) \neq p, \quad \pi(p) \neq 2p_0 - p, \quad \pi(p) \neq p_0 \quad \text{and} \quad \pi''(p) \neq 0. \tag{22}$$

We will assume throughout that $0 < p_0 < 1$. The other two cases are somewhat easier since some of the situations described below cannot occur. We will say a property holds "to the right (left) of p_0" if there exists an $\varepsilon > 0$ such that the property holds for all $p \in (p_0, p_0 + \varepsilon)$ (respectively, $(p_0 - \varepsilon, p_0)$). The proof that $P_n \Rightarrow B_{p_0}$ is split into various cases according to how many of the inequalities $\pi(p) > p$ and $\pi(p) > 2p_0 - p$ hold to the right and left of p_0. This gives nine cases, but two are excluded by differentiability of π at p_0 and two others are excluded by symmetry arguments.

We begin with the case for which $\pi(p) > 2p_0 - p$ for p to the left of p_0 and $\pi(p) < 2p_0 - p$ for p to the right of p_0. This case cannot occur without violating (21). To see this, observe that (21) is equivalent to the condition that the graphs of the function π and its inverse intersect only on the diagonal of $[0, 1] \times [0, 1]$, and then use the continuity of π.

Next, consider the case for which

$$p < \pi(p) < 2p_0 - p \tag{23}$$

to the left of p_0 and

$$2p_0 - p < \pi(p) < p \tag{24}$$

to the right of p_0. (This includes the case $|\pi'(p_0)| < 1$.) *Let* $\varepsilon > 0$. Suppose there is a sequence $r_k \to \infty$ such that

$$|p_0 - a(r_k)| \leqq \varepsilon \tag{25}$$

but

$$|p_0 - a(r_k + 1)| > \varepsilon \tag{26}$$

for all k. By Lemma 2, the sequence $\{P_{r_k}\}$ has a subsequence $\{P_{s_k}\}$ which converges to B_p where $|p - p_0| \leqq \varepsilon$. This implies $P_{s_{k+1}} \Rightarrow B_{\pi(p)}$ where $|\pi(p) - p_0| < \varepsilon$ provided ε is sufficiently small. In particular $|a(s_{k+1}) - p_0| < \varepsilon$ for k sufficiently large, which is a contradiction of (26). Since $|p_0 - a(n)| \leqq \varepsilon$ for infinitely many n, it follows that $|p_0 - a(n)| > \varepsilon$ for only finitely many n. Since ε is arbitrary, $a(n) \to p_0$. By Lemma 2, it follows that $P_n \Rightarrow B_{p_0}$ in this case.

We will make frequent use of the subsequence argument used above but will henceforth give fewer details.

The third case we consider is the case for which $\pi(p) < p$ to the left of p_0 and $\pi(p) > p$ to the right of p_0. If $P_n \not\Rightarrow B_{p_0}$, there must be infinitely many n such that $|a(n) - p_0| \leqq \varepsilon$ but $|a(n+1) - p_0| > \varepsilon$, provided $\varepsilon > 0$ is sufficiently small. We will show this leads to a contradiction. There are several subcases of which we single out one to give in detail. Assume that for a sequence of ε's decreasing to 0, there exist infinitely many n such that $|a(n) - p_0| \leqq \varepsilon$ but $a(n+1) < p_0 - \varepsilon$. Assume also that there exists a point $p_1 < p_0$ for which $\pi(p_1) = \pi(p_0)$. If more than one such p_1 exists, choose the largest value. By the continuity of π, it is easily seen via the interpretation of (21) in terms of the graph of π (given earlier) that $\pi(p) > p_1$ for $p_1 \leqq p \leqq p_0$. Let $\delta > 0$ be such that $\pi(p) > p_1 + \delta$ for all $p \in [p_1, p_0]$. Choose ε sufficiently small that $2\varepsilon + \delta < p_0 - p_1$, $\pi(p) < p_0 - 2\varepsilon$ for all $p \in (p_1 + \delta,\ p_0 - 2\varepsilon)$ and $\pi(p) < p$ for all $p \in [p_0 - 2\varepsilon, p_0)$. Some subsequence of $\{P_n\}$, with n as described above, must converge to some B_p and evidently $p_0 - \varepsilon \leqq p \leqq p_0$ while $p_1 + \delta < \pi(p) \leqq p_0 - \varepsilon$. It follows that there are infinitely many n such that $p_1 + \delta \leqq a(n) \leqq p_0 - \varepsilon$ but such that $a(n+1) \leqq p_1 + \delta$ or $a(n+1) \geqq p_0 - \varepsilon$. Taking a subsequence of $\{P_n\}$ for such n's gives a contradiction as in the previous case. We conclude that $P_n \Rightarrow B_{p_0}$ in this case also.

We next consider the case for which $\pi(p) < p$ to the left of p_0 and to the right of p_0. (This implies $\pi'(p_0) = 1$ and $\pi''(p) < 0$ near but not necessarily at p_0.) Suppose that for infinitely many n $|a(n) - p_0| \leqq \varepsilon$ but $|a(n+1) - p_0| > \varepsilon$. If $a(n+1) > p_0 + \varepsilon$ for infinitely many n, we get a contradiction as in the second case while if $a(n+1) < p_0 - \varepsilon$ for infinitely many n, we get a contradiction as in the third case. The case for which $\pi(p) > p$ to the left and right of p_0 is similar. Alternatively, it can be converted to the case just considered by switching 0's and 1's in Ω. This leads to the replacement of $\pi(p)$ by $1 - \pi(1-p)$, whose graph is obtained by rotating that of π by 180° about $(\frac{1}{2}, \frac{1}{2})$.

The final case, except for a similar symmetry argument, has (23) to the left of p_0 and $\pi(p) < 2p_0 - p$ to the right of p_0 (so that $\pi'(p_0) = -1$). Let $\varepsilon > 0$. Suppose there exist infinitely many n such that $|a(n) - p_0| < \varepsilon$ but $|a(n+1) - p_0| > 2\varepsilon$. If ε is small enough, this leads to a contradiction by the subsequence argument used

before. Now suppose there are infinitely many n such that $|a(n)-p_0|<\varepsilon$, $|a(n+1)-p_0|>\varepsilon$ and $|a(n+2)-p_0|>\varepsilon$. Choose a sequence $\{r_k\}$ of such n's for which $\{P_{r_k}\}$ converges, say to B_p where $|p-p_0|\leqq\varepsilon$. It is clear that $p>p_0$ and then that $\pi(p)<p_0$ and $\pi^2(p)>p_0$. By (21) and the continuity of π, it is easy to deduce that $\pi^2(p)<p$. By the subsequence arguments used before, this gives a contradiction. Therefore $|a(n)-p_0|<2\varepsilon$ except for finitely many n. It follows in this case also that $P_n \Rightarrow B_{p_0}$. This proves Theorem 2.

The assumption that (21) holds is essential. For suppose $\pi^2(p)=p$ but $\pi(p)\neq p$. Then take $P=B_p$ and observe that P_n oscillates with period 2. It follows from Theorem 2 that if (21) holds there cannot be any $p\in[0,1]$ such that $\pi^n(p)$ fails to converge as $n\to\infty$. This is a special case of results of Sharkovsky (1964, 1966). See also Stefan (1977). The assumption (21) can be relaxed by weakening the conclusion of the theorem, as follows.

Corollary 1. *Let $P\in\mathcal{M}$ and let T be a scaling transformation. Assume* (8) *holds. Further assume $\pi(p)\neq p$, $\pi(p)\neq 1-p$, and for some k*

$$\pi^{2^{k+1}}(p)=p \Leftrightarrow \pi^{2^k}(p)=p \tag{27}$$

for all $p\in[0,1]$. Then $\{P_n\}$ is asymptotically periodic with period 2^k. Specifically,

$$P_{2^k n+l} \Rightarrow B_{\pi^l(p)}$$

as $n\to\infty$, for $l=0,1,\ldots,2^k-1$, for some $p\in[0,1]$ for which $\pi^{2^k}(p)=p$.

Proof. Observe that the transformation $\hat{T}\equiv T^{2^k}$ is again a scaling transformation on Ω with scaling function $\hat{\varphi}$ and with corresponding function $\hat{\pi}$: $[0,1]\to[0,1]$ given by $\hat{\pi}=\pi^{2^k}$. The degree of $\hat{\pi}$ is at least that of π and as a result $\hat{\pi}$ is not the identity function. Also $\hat{\pi}$ satisfies (21) by (27). Finally, (8) must hold for at least one of the sequences $\{P_{n2^k+l}, n=1,2,\ldots\}$. The result holds for that value of l and hence for all l.

Condition (27) requires that there be no $p\in[0,1]$ such that the sequence $\{\pi^n(p)\}$ have (minimal) period 2^{k+1}. By the results of Sharkovsky (1964, 1966), this implies that the only possible periods for such sequences are $1,2,4,\ldots,2^k$. If there exists a $p\in[0,1]$ such that $\{\pi^n(p)\}$ has period not of the form 2^k for some k, then there also exists $q\in[0,1]$ such that $\{\pi^n(q)\}$ and hence $\{T^n B_q\}$ is not periodic or even asymptotically periodic. Examples of this situation can be constructed by taking r large and taking $\varphi(\omega_0,\omega_1,\ldots,\omega_{r-1})=1$ iff $\sum_{i=0}^{r-1}\omega_i$ is close to $\frac{1}{2}r$. (For such examples, the function π is the operating characteristic curve for a test of the hypothesis $p=\frac{1}{2}$ for a binomial distribution with the two-sided alternative $p\neq\frac{1}{2}$.)

If π is non-decreasing (21) holds automatically. If π is non-increasing, π^2 is non-decreasing so that (27) holds with $k=1$.

We do not know if the assumption $\pi(p)\neq p$ is essential. One type of scaling transformation which gives $\pi(p)\equiv p$ is obtained by taking $\varphi(\omega_0,\omega_1,\ldots,\omega_{r-1})=\omega_i$ for some $i\in\{0,1,\ldots,r-1\}$. In this case $P_n(\omega(0)=1)$ is constant as a function

of n so that $P_n \Rightarrow B_p$ where $p = P(\omega(0) = 1)$. The situation is less clear for other scaling transformations for which $\pi(p) \equiv p$; similar difficulties arise if $\pi(p) \equiv 1 - p$.

4. Majority and Related Transformations

Let r be a positive integer greater than one and let $k \in \{0, 1, \ldots, r\}$. Let $\varphi\colon \{0,1\}^r \to \{0,1\}$ be given by

$$\varphi(\omega_0, \omega_1, \ldots, \omega_{r-1}) = \begin{cases} 1 & \text{if } \sum_{i=0}^{r-1} \omega_i \geqq k \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

If k is taken to be $\frac{1}{2}(r+1)$ for odd r, the corresponding transformation is a majority transformation.

Theorem 3. *Let T be a scaling transformation with scaling function given by* (28). *Let $P \in \mathcal{M}$. Then P_n converges to a Bernoulli measure B_p where p is a fixed point of π (given by* (18)). *Also, P is invariant* iff $P = B_p$ *for such a p.*

Proof. We show Theorems 1 and 2 hold. First, observe that $\pi(p) \not\equiv p$. Also π is an increasing function of p so (21) holds. It remains to check (8), which we do by proving (9). Let $Q \in \mathcal{M}_2$. Let C be as in (6) and suppose $Q(C) = 1$. Then

$$Q(\{\omega\colon \omega(1) + \ldots + \omega(r-1) = k-1\}) = 1. \tag{29}$$

By shift-invariance

$$Q(\{\omega\colon \omega(0) + \omega(1) + \ldots + \omega(r-2) = k-1\}) = 1. \tag{30}$$

The combination of (29) and (30) implies

$$Q(\{\omega\colon \omega(0) = \omega(r-1)\}) = 1.$$

Thus Q is concentrated on the set of sequences of period $r-1$, which is impossible. This completes the proof.

Finding the fixed points of π is often difficult or impossible since it involves the solution of a polynomial equation which may be of high degree. In the case of majority transformations we do have an explicit solution.

Theorem 4. *Let T be a majority transformation. Then Theorem* 3 *holds and the fixed points of π are* $0, \frac{1}{2}$ *and* 1.

Proof. We need only solve the equation $\pi(p) = p$ for $0 \leqq p \leqq 1$. It is useful to indicate the dependence of π on r by writing $\pi = \pi_r$. We will show that for $k \geqq 1$,

$$\pi_{2k+1}(p) = p + \sum_{i=1}^{k} \binom{2i-1}{i} p^i (1-p)^i (2p-1). \tag{31}$$

First, for $k=1$, we have $\pi_3(p)=3p^2(1-p)+p^3$ which does reduce to (31). Assume we have shown the result for $k-1$. Then

$$\begin{aligned}
\pi_{2k+1}(p) &= \sum_{i=k+1}^{2k+1} \binom{2k+1}{i} p^i(1-p)^{2k+1-i} \\
&= \sum_{i=k+1}^{2k+1} \left\{ \binom{2k-1}{i} + 2\binom{2k-1}{i-1} + \binom{2k-1}{i-2} \right\} p^i(1-p)^{2k+1-i} \\
&= \sum_{i=k+1}^{2k-1} \binom{2k-1}{i} p^i(1-p)^{2k+1-i} + 2\sum_{i=k}^{2k-1} \binom{2k-1}{i} p^{i+1}(1-p)^{2k-i} \\
&\quad + \sum_{i=k-1}^{2k-1} \binom{2k-1}{i} p^{i+2}(1-p)^{2k-i-1} \\
&= \sum_{i=k}^{2k-1} \binom{2k-1}{i} p^i(1-p)^{2k-1-i}[(1-p)^2+2p(1-p)+p^2] \\
&\quad - \binom{2k-1}{k} p^k(1-p)^{k+1} + \binom{2k-1}{k-1} p^{k+1}(1-p)^k \\
&= \pi_{2k-1}(p) + \binom{2k-1}{k} p^k(1-p)^k(2p-1).
\end{aligned}$$

Thus (31) holds for all $k \geqq 1$. The terms on the right side of (31), except p itself, are zero for $p=0, \frac{1}{2}$ or 1. For $0<p<\frac{1}{2}$ they are all negative and for $\frac{1}{2}<p<1$ they are all positive. The theorem follows from these facts.

The notion of majority only makes sense for odd values of r. We generalize the notion to even r as follows. There are an even number of r-tuples in $\{0,1\}^r$ which have exactly $\frac{1}{2}r$ ones and $\frac{1}{2}r$ zeros (since they can be paired by switching zeros and ones). Select half them and define $\varphi(\omega_0, \omega_1, \ldots, \omega_{r-1})=1$ iff either $\sum \omega_i > \frac{1}{2}r$ or $(\omega_0, \ldots, \omega_{r-1})$ is one of the selected r-tuples (in the case $\sum \omega_i = \frac{1}{2}r$). By definition the function $\pi=\pi_r$ is given by

$$\pi_r(p) = \tfrac{1}{2}\binom{t}{\frac{1}{2}r} p^{\frac{1}{2}r}(1-p)^{\frac{1}{2}r} + \sum_{i=\frac{1}{2}r+1}^{r} \binom{r}{i} p^i(1-p)^{r-i}.$$

Consider the new scaling function $\hat{\varphi}$ defined by $\hat{\varphi}(\omega_0, \ldots, \omega_{r-1})=1$ iff $\sum \omega_i > \frac{1}{2}r$ or $\omega_{r-1}=0$ and $\sum \omega_i = \frac{1}{2}r$. Then $\hat{\varphi}(\omega_0, \ldots, \omega_{r-1})=1$ iff the majority of $\omega_0, \omega_1, \ldots, \omega_{r-2}$ are 1's. It is therefore clear that

$$\pi_r(p) = \hat{\pi}_r(p) = \pi_{r-1}(p).$$

To obtain the analogue of Theorem 4 for r even, we need only check that (9) holds for all $Q \in \mathscr{M}_2$. Let us suppose $\varphi(0,1,0,1,\ldots,0,1)=1$, the other case being similar. Then

$$\begin{aligned}
1-Q(C) &\geqq Q(\{\omega\colon \omega(1)=\omega(3)=\ldots=\omega(r-1)=1\}) \\
&= [Q(\{\omega\colon \omega(0)=1\})]^{\frac{1}{2}r} > 0
\end{aligned}$$

unless $Q(\{\omega: \omega(0)=1\})=0$. In the latter case,

$$1-Q(C)\geqq Q(\{\omega: \omega(0)=\omega(1)=\ldots=\omega(r-1)=0\})=1.$$

Thus, Theorem 4 extends to these "even majority" transformations.

5. Which Limit is Obtained?

In cases where Theorem 2 applies, there may be several fixed points for the function π. We do not have very good methods for deciding which limit is obtained for a given P.

If $P=B_p$, $\lim P_n=B_{\lim \pi^n(p)}$, so the problem can be solved by determining $\lim \pi^n(p)$. This can be done (in principle) by examining each fixed point and observing whether it is "attractive" or "repulsive", as determined by the behaviour of π near the fixed point. Suppose for example that T is a majority transformation. It is easy to see that $\pi^n(p)$ converges to 0, $\frac{1}{2}$ or 1 according as $p<\frac{1}{2}$, $p=\frac{1}{2}$ or $p>\frac{1}{2}$ (that is, 0 and 1 are attractive and $\frac{1}{2}$ is repulsive).

If P is not Bernoulli, the problem is much more complicated. Take for T the majority transformation with scale $r=3$. A sufficient condition for $P_n \Rightarrow B_{\frac{1}{2}}$ is that P be a symmetric measure. A sufficient condition for $P_n \Rightarrow B_0$ (or $P_n \Rightarrow B_1$) is that P be stochastically less (respectively, greater) than B_p for some $p<\frac{1}{2}$ (resp. $>\frac{1}{2}$). See for example Kamae, Krengel and O'Brien (1977).

One might conjecture that $P_n \Rightarrow B_0$, $B_{\frac{1}{2}}$ or B_1 according as $P(\omega(0)=1)<\frac{1}{2}$, $=\frac{1}{2}$ or $>\frac{1}{2}$ since this is the case if P is Bernoulli. The following example shows this to be false. Let $\{X_n\}$ be a stationary Markov chain with states 1, 2, 3 and 4 and transition matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Then $P(X_n=1)=P(X_n=4)=\frac{1}{6}$ while $P(X_n=2)=P(X_n=3)=\frac{1}{3}$ for all n. Let $f(1)=f(2)=0$ and $f(3)=f(4)=1$. The distribution of $\{f(X_n)\}$ has the property that $P(\omega(i)=1)=\frac{1}{2}$ while $P_1(\omega(i)=1)=\frac{7}{12}$. A slight perturbation of the above matrix gives an example for which $P(\omega(0)=1)<\frac{1}{2}$ but $P_1(\omega(0)=1)>\frac{1}{2}$. At least one, and by symmetry both, of the statements

$$P_n \Rightarrow B_0 \quad \text{if } P(\omega(0)=1)<\tfrac{1}{2}$$

and

$$P_n \Rightarrow B_1 \quad \text{if } P(\omega(0)=1)>\tfrac{1}{2}$$

must be false.

6. Measures Outside $\mathscr{M}$

It is not the case that the only measures which are invariant with respect to a scaling transformation are Bernoulli measures. We demonstrate this by giving several examples in the case when T is the majority transformation with scale 3.

First, let P be the measure which puts an atom of $\frac{1}{2}$ at each of two sequences of alternating zeros and ones. Then P is S-invariant and T-invariant. Moreover, $P(\{\omega\colon T\omega=\omega\})=1$. It is clear that S is ergodic with respect to P although the tail field is non-trivial. On the other hand, T is not ergodic with respect to P. Since

$$|P(\omega(0)=\omega(n)=1)-P(\omega(0)=1)\,P(\omega(n)=1)|=\tfrac{1}{4}$$

for all n, P is not strongly mixing. On the other hand, if P puts unit mass on one of the two alternating sequences, then P is strongly mixing and T-invariant and T is ergodic with respect to P, but P is not S-invariant. A similar example is obtained by letting P have an atom of size $\frac{1}{4}$ on each of the four sequences $\ldots 01100110011\ldots$.

There is also at least one S- and T-invariant measure which has no atoms. Let Q assign values in such a way that

$$Q(\omega(2i)=0,\,\omega(2i+1)=1)=Q(\omega(2i)=1,\,\omega(2i+1)=0)=\tfrac{1}{2}$$

and such that all pairs $(\omega(2i),\omega(2i+1))$ are independent. Let $P=\frac{1}{2}(Q+SQ)$. Then P is S- and T-invariant. Again, S is ergodic with respect to P but T is not ergodic, and P is not strongly mixing.

7. Sum Modulo Two Transformations

Let T be the transformation obtained by defining φ as in (4). We observed earlier that Lemma 2 does not apply to T. We now investigate when the conclusions of Theorems 1 and 2 hold for T.

As a first step, we examine the function π which we denote here by π_r, where r is the scale. It is easily seen that

$$\pi_r(p)=p+(1-2p)\,\pi_{r-1}(p). \tag{32}$$

It follows from (32) and the fact that $0<\pi_{r-1}(p)<1$ for $p\in(0,1)$ that the fixed points of π_r are $0, \frac{1}{2}$ and, for even r, 1. It also follows from (32) and induction that the only solution of $\pi_r(p)=\frac{1}{2}$ is $p=\frac{1}{2}$. This implies that (21) holds.

Let $P\in\mathscr{M}$ be such that $\{P_n\}$ converges to a Bernoulli measure B_p. This assumption eliminates the need for Lemma 2. It therefore follows from Theorem 2 that p is one of the fixed points of π_r, namely $0, \frac{1}{2}$ or 1. It follows that

$$P\left(\sum_{i=0}^{r^n-1}\omega(i)\equiv 1(\mathrm{mod}\,2)\right)=P_n(\omega(0)=1)\to 0, \tfrac{1}{2}\text{ or }1. \tag{33}$$

This fact can be viewed as a local limit theorem for sums modulo 2.

We will next show that for some very simple measures $P \in \mathcal{M}$, (33) fails and hence $\{P_n\}$ cannot converge to a Bernoulli limit. Let $\{X_n, n \in Z\}$ be a stationary Markov chain on a probability space $(\Omega', \mathfrak{F}', P')$ with state space $\{0, 1, 2, 3\}$ and with transition matrix A given by

$$\begin{array}{c} \\ (0) \\ (1) \\ (2) \\ (3) \end{array} \begin{array}{c} \begin{array}{cccc} (0) & (1) & (2) & (3) \end{array} \\ \begin{bmatrix} \alpha & 0 & 0 & 1-\alpha \\ \alpha & 0 & 0 & 1-\alpha \\ 0 & \alpha & 1-\alpha & 0 \\ 0 & \alpha & 1-\alpha & 0 \end{bmatrix} \end{array}$$

where $0 < \alpha < 1$. Let $f: \{0, 1, 2, 3\} \to \{0, 1\}$ be given by $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$. Let P be the distribution of $\{f(X_n)\}$. Then P is Markov-based and, since the rows of A^2 are all given by the stationary measure $(\alpha^2, \alpha(1-\alpha), (1-\alpha)^2, \alpha(1-\alpha))$, $P \in \mathcal{M}_2$.

Let $S_n = \sum_{i=0}^{n} f(X_i) \pmod 2$. It is easily seen that $\{(X_n, S_n), n = 0, 1, \ldots\}$ is also a positive recurrent Markov chain. Its eight point state space has $I_1 = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$ and $I_2 = \{(0, 1), (1, 1), (2, 0), (3, 0)\}$ as its closed irreducible classes. Furthermore (X_0, S_0) must be one of $(0, 0)$, $(1, 1)$, $(2, 0)$ or $(3, 1)$ with probability α^2, $\alpha(1-\alpha)$, $(1-\alpha)^2$, $\alpha(1-\alpha)$ respectively. Thus $P'((X_0, S_0) \in I_1) = \alpha$ and $P'((X_0, S_0) \in I_2) = 1 - \alpha$. The usual limit theorem for Markov chains now shows $P'(S_n = 1) \to 2\alpha(1-\alpha)$. It follows that $\{P_n\}$ can only converge to a Bernoulli measure if $\alpha = \frac{1}{2}$. Actually, it can be shown directly that P is invariant under T for all $\alpha \in (0, 1)$.

We next show that for "most" Markov-based $P \in \mathcal{M}$, it is the case that $\{P_n\}$ converges to $B_{\frac{1}{2}}$. Suppose the Markov chain $\{X_n\}$ has the property that there exist states $i_0, i_1, \ldots, i_k$ and $j_0 = i_0, j_1, j_2, \ldots, j_k = i_k$ such that $P'(X_n = i_h | X_{n-1} = i_{h-1}) > 0$ and $P'(X_n = j_h | X_{n-1} = j_{h-1}) > 0$ for $h = 1, 2, \ldots, k$ and such that the number of h's for which $f(i_h) \neq f(j_h)$ is odd. It is easily seen that this is equivalent to the Markov chain $\{(X_n, S_n)\}$ being irreducible and aperiodic, which in turn implies $P'(S_n = 1 | X_0 = x) \to \frac{1}{2}$ for all x. Then, for example, $P_n(\omega(0) = 1, \omega(1) = 1) = P'(S_{r^{n-1}} = 1, S_{r^{2n-1}} = 0) \to \frac{1}{4}$, so that P_n has a Bernoulli limit.

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A Construction of Diffusion Processes with Singular Product Measures*

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Introduction

Let Ω be a domain in $R^d (d \geqq 2)$ and put $\Omega_d = \{x_d \in R^1 : (x_1, \ldots, x_{d-1}, x_d) \in \Omega$ for some $(x_1, \ldots, x_{d-1}) \in R^{d-1}\}$. m denotes a nonnegative Radon measure on Ω_d, i.e. $m(I)$ is finite for every compact set I of Ω_d. Let $m(dt) = m_0(t)\,dt + m_1(dt)$ denote the Lebesgue decomposition. Namely, dt is the one-dimensional Lebesgue measure and $m_1(dt)$ is singular with respect to dt. We assume the following:

(1) $\inf\{m_0(t) : t \in U\} > 0$ for every bounded open set U with $\overline{U} \subset \Omega_d$.

Let us define positive Radon measures $\nu_{ij}(dx) = \nu_{ij}(dx_1 \ldots dx_d)$, $1 \leqq i \leqq d$ on Ω by

$$\nu_{ij}(dx) = \begin{cases} \nu(dx) \equiv dx_1 \ldots dx_{d-1}\, m(dx_d) & \text{if } 1 \leqq i, j \leqq d-1, \\ dx \equiv dx_1 \ldots dx_{d-1}\, dx_d & \text{otherwise.} \end{cases} \tag{2}$$

Consider the functions $a_{ij}, 1 \leqq i, j \leqq d$ on Ω having the properties:

(3) For each i, j $a_{ij} = a_{ji}$ and a_{ij} is of C^{d_0}-class on Ω, where $d_0 = [(d-1)/2] + 1$. Moreover for any compact set $K \subset \Omega$ there is a positive constant $\gamma = \gamma(K)$ such that

$$\gamma^{-1} \sum_{i=1}^{d} \xi_i^2\, \nu_{ii}(dx) \leqq \sum_{i,j=1}^{d} \xi_i \xi_j a_{ij}(x)\, \nu_{ij}(dx) \leqq \gamma \sum_{i=1}^{d} \xi_i^2\, \nu_{ii}(dx) \qquad x \in K,\ \xi \in R^d.$$

Now we shall introduce the bilinear form $\mathcal{Q}$:

$$\begin{aligned} &\mathcal{D}[\mathcal{Q}] = C_0^\infty(\Omega) \\ &\mathcal{Q}(u, v) = \sum_{i,j=1}^{d} \int_\Omega \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, a_{ij}(x)\, \nu_{ij}(dx). \end{aligned} \tag{4}$$

Here $C_0^\infty(\Omega)$ = the space of C^∞-functions with compact supports in Ω. As will be seen later, $\mathcal{Q}$ is closable on $L^2(\Omega; dx)$ with inner product $(\ ,\ )_\Omega$. Therefore by using M. Fukushima's results [4] we obtain a diffusion except for a set of zero

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capacity whose Dirichlet form coincides with the smallest closed extension $\mathscr{E}$ of $\mathscr{Q}$. In this paper we get however a stronger result:

Theorem. *There exists a unique diffusion process on Ω, whose resolvent R_λ satisfies the following: Let $\lambda>0$. (i) $R_\lambda f$ is continuous on each compact set in Ω for every $f\in L^2(\Omega;dx)\cap L^p(\Omega;dx)$ with $p>d$. (ii) $R_\lambda f$ is a unique solution of*

$$\begin{aligned} &R_\lambda f\in\mathscr{D}[\mathscr{E}] \\ &\mathscr{E}(R_\lambda f,\varphi)+\lambda(R_\lambda f,\varphi)_\Omega=(f,\varphi)_\Omega \quad \varphi\in C_0^\infty(\Omega), \end{aligned} \tag{5}$$

for every $f\in L^2(\Omega;dx)$.

A probabilistic approach due to N. Ikeda and S. Watanabe [9] leads us to a nice diffusion in the case where $m(dt)=dt+\delta_{\{0\}}(dt)$ and $a_{ij}=\delta_{ij}$. On the other hand, following the arguments of M. Kanda [11] and H. Kunita [12], who get diffusions corresponding to $m(dt)=dt$, the author [16] constructed a diffusion associated with $\mathscr{E}$ for $m_0\equiv 1$ and $m_1\equiv$ a sum of δ-measures. A number of analytical preparations were made in [16]. Our theorem will be proved along the same lines.

First we construct diffusions on R^d under additional restrictions on m and a_{ij}. In Sect. 1 some function spaces relevant to v are specified and their properties are studied. In the first half of Sect. 2 $\{G_\lambda;\lambda>0\}$ is defined by solutions of (5). It will be then shown that G_λ maps C_0 into C_∞ densely, where C_0 [resp. C_∞]=the space of continuous functions having compact supports [resp. vanishing at infinity]. To this end auxiliary lemmas are needed. The major part of their proofs is closely follows arguments about elliptic partial differential equations which are due to L. Nirenberg [13] and G. Stampacchia [14]. In the last half, H. Kunita's method [12] is used to construct diffusions on R^d. In Sect. 3 we prove the theorem by employing M. Fukushima's result [6]. Section 4 is devoted to the study of polar sets. As will be seen, any polar set of our diffusion is also polar with respect to the Brownian motion, but the converse is false.

1. Preliminaries

Throughout this section we suppose that m is defined on R^1 and satisfies

$$A\equiv\inf\{m_0(t):t\in R^1\}>0, \qquad M\equiv\sup\{m([i,i+1]):i=0,\pm1,\ldots\}<\infty. \tag{6}$$

v and v_{ij}'s are defined on R^d by (2).

Function Spaces. Let Ω be an arbitrary domain in R^d. We shall discuss the weak differentiability of functions on Ω. We use the notation: $D_i=\partial/\partial x_i$ for $1\leqq i\leqq d$, $D_{x'}^k=D_1^{k_1}\ldots D_{d-1}^{k_{d-1}}$ for an $d-1$ tuple $k=(k_1,\ldots,k_{d-1})$ of nonnegative integers, and $|k|=k_1+\ldots+k_{d-1}$. Let $u\in L^1_{loc}(\Omega;dv)$ and $|k|\geqq 1$. If there exists a function $v\in L^1_{loc}(\Omega;dv)$ such that

$$\int_\Omega u D_{x'}^k\varphi\,dv=(-1)^{|k|}\int_\Omega v\varphi\,dv \qquad \varphi\in C_0^\infty(\Omega),$$

then such v is determined uniquely up to sets of v-measure zero and it is denoted by $\mathbf{D}^k_{x^i} u$. We call them the weak derivatives (with respect to v) of u. For $1 \leqq i \leqq d$ and $u \in L^1_{loc}(\Omega; dx)$ $D_i u$ stands for the distributional derivative, that is, $D_i u$ belongs to $L^1_{loc}(\Omega; dx)$ and satisfies

$$\int_\Omega u D_i \varphi \, dx = -\int_\Omega D_i u \, \varphi \, dx \qquad \varphi \in C_0^\infty(\Omega).$$

$D^k_{x^i} u$, $D^k u$ etc. are defined in the same way.

For any $1 < p < \infty$ we introduce the space $\mathscr{W}^p(\Omega; v)$:

$$\mathscr{W}^p(\Omega; v) = \{u \in L^p(\Omega; dv) : \mathbf{D}_i u \in L^p(\Omega; dv),\ 1 \leqq i \leqq d-1,\ D_d u \in L^p(\Omega; dx)\}.$$

By the same method as used in Sobolev spaces we see that: $u \in L^p(\Omega; dv)$ belongs to $\mathscr{W}^p(\Omega; v)$ if and only if for each $1 \leqq i \leqq d$ there is a version u^i such that $u = u^i$ v_{ii}-a.e. and u^i is absolutely continuous (with respect to the one-dimensional Lebesgue measure) on $\Omega_i(x)$ for $\hat{v}_i$-a.e. $\hat{x}_i$, and its (usual) derivative $\partial u^i / \partial x_i$ (which exists v_{ii}-a.e.) belongs to $L^p(\Omega; dv_{ii})$. Here $x = (x_1, \ldots, x_d)$, $\hat{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$, $\Omega_i(x)$ is the intersection of Ω with the line through x parallel to the x_i coordinate axis, and $\hat{v}_i(d\hat{x}_i) = dx_1 \ldots dx_{i-1} dx_{i+1} \ldots m(dx_d)$. In particular $\hat{v}_d(d\hat{x}_d) = dx_1 \ldots dx_{d-1}$. Since $\mathscr{W}^p(\Omega; v)$ is complete with respect to the norm $|||\cdot|||_{p, \Omega, v}$:

$$|||u|||_{p, \Omega, v} = \sum_{i=1}^{d-1} \left(\int_\Omega |\mathbf{D}_i u|^p \, dv\right)^{1/p} + \left(\int_\Omega |D_d u|^p \, dx\right)^{1/p} + \left(\int_\Omega |u|^p \, dv\right)^{1/p}, \tag{7}$$

we can define the following spaces:

$$\mathscr{F}^p(\Omega; v) = \text{the closure of } \{u \in C^1(\overline{\Omega}) : |||u|||_{p, \Omega, v} < \infty\} \text{ in } \mathscr{W}^p(\Omega; v),$$

$$\mathscr{F}_0^p(\Omega; v) = \text{the closure of } C_0^\infty(\Omega) \text{ in } \mathscr{W}^p(\Omega; v),$$

where $C^1(\overline{\Omega}) =$ the space of functions of C^1-class on $\overline{\Omega}$. In situations where no confusion of domains and measures occurs, $|||\cdot|||_{p, \Omega, v}$, $\mathscr{F}^p(\Omega; v)$ and $\mathscr{F}_0^p(\Omega; v)$ are written as $|||\cdot|||_p$, $\mathscr{F}^p$ and $\mathscr{F}_0^p$, respectively.

We note some of the simple properties of these spaces. Let $H(a, h) = \{x \in R^d : |x_i - a_i| < h,\ 1 \leqq i \leqq d\}$ with $a = (a_1, \ldots, a_d)$ and $h > 0$.

Proposition 1. *(i) A function $u \in \mathscr{W}^p(R^d; v)$ belongs to $\mathscr{F}_0^p(R^d; v)$ provided $D_d \mathbf{D}_i u \in L^p(R^d; dx)$, $1 \leqq i \leqq d-1$. (ii) Put $H = H(a, h)$. Every function u of $\mathscr{F}^p(H; v)$ has a version v such that (a) $v = u$ v-a.e., (b) v is absolutely continuous on $H_d(x)$ for $\hat{v}_d$-a.e. $\hat{x}_d$, and (c) $D_d v = D_d u$ a.e. (iii) The norm $|||\cdot|||_{p, H, v}$ is equivalent with the norm $|\cdot|_{p, H, v}$ in $\mathscr{F}^p(H; v)$, where*

$$|u|_{p, H, v} = \sum_{i=1}^{d-1} \left(\int_H |\mathbf{D}_i u|^p \, dv\right)^{1/p} + \left(\int_H |D_d u|^p \, dx\right)^{1/p} + \left(\int_H |u|^p \, dx\right)^{1/p}. \tag{8}$$

(iv) If $\mathscr{F}^p$ is replaced by $\mathscr{F}_0^p$ in (iii), the assertion is valid for arbitrary domains.

Proof. (i) We shall show that u can be approximated in $\mathscr{F}_0^p$ by functions belonging to $C_0^\infty(R^d)$. We may assume that Supp$[u]$ is compact without loss of

generality. Let J_ε, $\varepsilon>0$ be mollifiers and set

$$u_\varepsilon(x)=J_\varepsilon * u(x)=\int_{R^d} J_\varepsilon(x-y)\,u(y)\,dy\in C_0^\infty(R^d).$$

Hölder's inequality leads us to

$$\begin{aligned}
&\int_{R^d}|u(x)-u_\varepsilon(x)|^p\,v(dx)\\
&\quad\leqq\int_{R^d}v(dx)\int_{R^d}J_\varepsilon(y)|u(x-y)-u(x)|^p\,dy\\
&\quad\leqq 2^{p-1}\Big\{\int_{R^d}J_\varepsilon(y)\,dy\int_{R^1}m(dx_d)\int_{R^{d-1}}|u(x'-y',x_d)-u(x',x_d)|^p\,dx'\\
&\qquad+\int_{R^d}J_\varepsilon(y)\,dy\int_{R^d}v(dx)\left|\int_{x_d}^{x_d-y_d}|D_d u(x'-y',t)|^p\,dt\right||y_d|^{p-1}\Big\}\\
&\quad\leqq 2^{p-1}\Big\{\sup_{|y'|<\varepsilon}\int_{R^1}m(dx_d)\int_{R^{d-1}}|u(x'-y',x_d)-u(x',x_d)|^p\,dx'\\
&\qquad+\varepsilon^{p-1}\,m(K_\varepsilon)\int_{R^d}|D_d u(x)|^p\,dx\Big\}\\
&\quad\to 0\quad\text{as }\varepsilon\downarrow 0,
\end{aligned}$$

where $x=(x',x_d)$ and K_ε denotes the x_d-projection of ε-neighborhood of $\mathrm{Supp}[u]$: $K_\varepsilon=\{t\in R^1:|t-x_d|<\varepsilon,\ (x',x_d)\in\mathrm{Supp}[u]\text{ for some }x_d\in R^1,x'\in R^{d-1}\}$. In the same way

$$\sum_{i=1}^{d-1}\|\mathbf{D}_i u-\mathbf{D}_i u_\varepsilon;\,L^p(R^d;dv)\|\to 0\quad\text{as }\varepsilon\downarrow 0,^1$$

because $\mathbf{D}_i u_\varepsilon=J_\varepsilon*\mathbf{D}_i u$ in the weak sense and $D_i\mathbf{D}_i u$ belongs to $L^p(R^d;dx)$. Obviously $\|D_d u-D_d u_\varepsilon;\,L^p(R^d;dx)\|\to 0$ as $\varepsilon\downarrow 0$, so that $|||u-u_\varepsilon|||_p\to 0$. Thus we get $u\in\mathscr{F}_0^p(R^d;v)$.

(ii) We observe the inequality

$$\int_H|u|^p\,dv\leqq 2^{p-1}\,m([a_d-h,a_d+h])\left\{\frac{1}{2h}\int_H|u|^p\,dx+(2h)^{p-1}\int_H|D_d u|^p\,dx\right\},\tag{9}$$

for $u\in C^1(\overline{H})$, which is derived from the proof of [1; Lemma 5.7]. For $u\in\mathscr{F}^p(H;v)$ we now have a version v satisfying $v=u$ a.e. instead of (a), and besides (b), (c). But by (9) this v satifies (a).

The assertion of (iii) immediately follows from (9).

(iv) It suffices to note $R^d=\bigcup_a\overline{H(a,1/2)}$, where the union is taken over the set of all d-tuples a of integers. Indeed, by (9)

$$\int_{R^d}|u|^p\,dv\leqq 2^{p-1}\,M\Big\{\int_{R^d}|u|^p\,dx+\int_{R^d}|D_d u|^p\,dx\Big\},\qquad u\in C_0^\infty(R^d).\tag{10}$$

q.e.d.

[1] In a normed space X the norm is denoted by $\|\cdot;X\|$ except where simpler notations are introduced

Imbeddings. Following Gagliardo's arguments [7] (cf. [1; Theorem 5.4]), we now show some imbedding characteristics similar to the well-known Sobolev imbedding theorem. Let $H=H(a,h)$ be a cube as mentioned above.

Proposition 2. Let $1<p<d$, $p\leqq q\leqq dp/(d-p)$ or $p=d$, $p\leqq q<\infty$. Then

$$\|u;L^q(H;dv)\|\leqq C_1(d,p,q,A)\,v(H)^{1/q-1/p+1/d}\|u\|_{p,H,v}\qquad u\in\mathscr{F}_0^p(H;v), \tag{11}$$

where

$$\|u\|_{p,H,v}=\sum_{i=1}^{d-1}\Big(\int_H|\mathbf{D}_i u|^p\,dv\Big)^{1/p}+\Big(\int_H|D_d u|^p\,dx\Big)^{1/p}.$$

Furthermore

$$\|u;L^q(R^d;dv)\|\leqq C_2(d,p,q,A,M)\|\!|u|\!\|_{p,R^d,v}\qquad u\in\mathscr{F}_0^p(R^d;v). \tag{12}$$

Proof. It is sufficient to establish

$$\|u;L^{q_0}(H;dv)\|\leqq c_1(d,p,A)\left(\sum_{i=1}^d\int_H|D_i u|^p\,dv_{ii}\right)^{1/p}\qquad u\in C_0^\infty(H), \tag{13}$$

$$\|u;L^{q_0}(H;dv)\|\leqq c_2(d,p,A)\left(\sum_{i=1}^d\int_H|D_i u|^p\,dv_{ii}+\int_H|u|^p\,dv\right)^{1/p}\qquad u\in C^1(\overline{H}), \tag{14}$$

where $p<d$ and $q_0=dp/(d-p)$. Indeed, in the same way as [1; Lemma 5.12, Corollary 5.13, Lemma 5.14] we can see that (13) and (14) combined with Hölder's inequality conclude (11) and (12) respectively.

We now assume $p<d$ and $u\in C_0^\infty(H)$. e_i denotes a unit vector along the x_i-axis. Put $\beta=p(d-1)/(d-p)$ $(\geqq 1)$ and $H_i=(a_i-h,\,a_i+h)$.

$$|u(x)|^\beta\leqq\int_0^{2h}\left|\frac{d}{dt}|u(x+t\,e_i)|^\beta\right|dt\leqq\beta\int_{H_i}|u|^{\beta-1}|D_i u|\,dx_i. \tag{15}$$

If $\hat{H}_i=\{\hat{x}_i=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)\colon(x_1,\dots,x_d)\in H$ for some $x_i\in R^1\}$ and $G_i(\hat{x}_i)=\sup\{|u(y)|^{p/(d-p)}\colon y\in H_i(x)\}$, then

$$\begin{aligned}\int_{\hat{H}_i}G_i(\hat{x}_i)^{d-1}\,\hat{v}_i(d\hat{x}_i)&\leqq\int_{\hat{H}_i}\hat{v}_i(d\hat{x}_i)\int_{H_i}\beta|u|^{\beta-1}|D_i u|\,dx_i\\&=\beta\int_H|u|^{\beta-1}|D_i u|\,dv_{ii}\leqq\beta\Big(\int_H|u|^{(\beta-1)p'}\,dv_{ii}\Big)^{1/p'}\Big(\int_H|D_i u|^p\,dv_{ii}\Big)^{1/p}\\&\leqq\beta\Big(\int_H|u|^{q_0}\,dv_{ii}\Big)^{1/p'}\left(\sum_{i=1}^d\int_H|D_i u|^p\,dv_{ii}\right)^{1/p},\end{aligned}$$

where $p'=p/(p-1)$. On the other hand, in case of $d>2$ Hölder's inequality gives

$$\begin{aligned}\int_H|u|^{q_0}\,dv&\leqq\int_H\prod_{i=1}^d G_i(\hat{x}_i)\,v(dx)=\int_{H_d}m(dx_d)\int_{\hat{H}_d}\prod_{i=1}^{d-1}G_i(\hat{x}_i)\,G_d(\hat{x}_d)\,d\hat{x}_d\\&\leqq\int_{H_d}m(dx_d)\left(\int_{\hat{H}_d}\prod_{i=1}^{d-1}G_i(\hat{x}_i)^s\,d\hat{x}_d\right)^{1/s}\Big(\int_{\hat{H}_d}G_d(\hat{x}_d)^{d-1}\,d\hat{x}_d\Big)^{1/(d-1)}\end{aligned}$$

with $s=(d-1)'=(d-1)/(d-2)$. Then Gagliardo's lemma [7] (cf. [1; Lemma 5.9]) leads us to

$$\left(\int_{\hat{H}_d}\prod_{i=1}^{d-1} G_i(\hat{x}_i)^s\, d\hat{x}_d\right)^{d-2} \leqq \prod_{i=1}^{d-1}\int_{\hat{H}_{id}} G_i(\hat{x}_i)^{s(d-2)}\, d\hat{x}_{id} = \prod_{i=1}^{d-1}\int_{\hat{H}_{id}} G_i(\hat{x}_i)^{d-1}\, d\hat{x}_{id},$$

where $\hat{H}_{id}=\{\hat{x}_{id}=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{d-1}): (x_1,\ldots,x_d)\in H \text{ for some } x_i, x_d\in R^1\}$. Therefore using Hölder's inequality again, we find

$$\begin{aligned}\int_H |u|^{q_0}\, dv &\leqq \int_{H_d} m(dx_d)\left\{\left(\prod_{i=1}^{d-1}\int_{\hat{H}_{id}} G_i(\hat{x}_i)^{d-1}\, d\hat{x}_{id}\right)\left(\int_{\hat{H}_d} G_d(\hat{x}_d)^{d-1}\, d\hat{x}_d\right)\right\}^{1/(d-1)} \\ &\leqq \prod_{i=1}^{d-1}\left(\int_{H_d} m(dx_d)\int_{\hat{H}_{id}} G_i(\hat{x}_i)^{d-1}\, d\hat{x}_{id}\right)^{1/(d-1)}\left(\int_{\hat{H}_d} G_d(\hat{x}_d)^{d-1}\, d\hat{x}_d\right)^{1/(d-1)} \\ &= \prod_{i=1}^{d}\left(\int_{\hat{H}_i} G_i(\hat{x}_i)^{d-1}\, \hat{v}_i(d\hat{x}_i)\right)^{1/(d-1)},\end{aligned}$$

which holds in case of $d=2$, too. Thus

$$\begin{aligned}\int_H |u|^{q_0}\, dv &\leqq \prod_{i=1}^{d}\left\{\beta\left(\int_H |u|^{q_0}\, dv_{ii}\right)^{1/p'}\left(\sum_{i=1}^{d}\int_H |D_i u|^p\, dv_{ii}\right)^{1/p}\right\}^{1/(d-1)} \\ &\leqq \left\{\beta\left(\int_H |u|^{q_0}\, dv\right)^{1/p'}\left(\sum_{i=1}^{d}\int_H |D_i u|^p\, dv_{ii}\right)^{1/p}\right\}^{d/(d-1)} (1/A)^{1/p'(d-1)}.\end{aligned}$$

By cancellation we obtain (13). In order to get (14), it suffices to use [1; Lemma 5.10, (19)] instead of (15). q.e.d.

Next we are concerned with the imbedding of $\mathscr{F}^p(H;v)$ into $C^\lambda(\overline{H})$:=the space of Hölder continuous functions on H of exponent λ. This can not be expected as long as we do not require the functions of $\mathscr{F}^p(H;v)$ to satisfy some additional restrictions. It is known that all functions u which, together with every distributional derivative $D^k u$ of order $|k|\leqq s$, belongs to $L^p(H;dx)$ are imbedded into $C^\lambda(\overline{H})$, where $0<\lambda\leqq s-d/p$ if $(s-1)p<d<sp$ or $0<\lambda<1$ if $d=(s-1)p$. We turn this imbedding into a convenient form for our application carried out in the next section.

Before we begin the proof, we introduce a new notation. If $u\in L^1_{loc}(\Omega;dx)$ satisfies

$$\int_\Omega v\varphi\, dv = -\int_\Omega u D_d\varphi\, dx \qquad \varphi\in C_0^\infty(\Omega)$$

for some $v\in L^1_{loc}(\Omega;dv)$, then there is a version $\tilde{u}$ such that $\tilde{u}=u$ a.e., $\tilde{u}$ is absolutely continuous with respect to m on $\Omega_d(x)$ for $\hat{v}_d-$a.e. $\hat{x}_d$ and the Radon-Nikodym derivative coincides with v v-a.e. Thus we denote such (uniquely determined, up to sets of v-measure zero) v by $\tilde{D}_d u$.

Proposition 3. *Let $u\in\mathscr{F}^p(H;v)$ be such that $D_x^k u\in\mathscr{F}^p(H;v)$, $0\leqq|k|\leqq s$ and $\tilde{D}_d D_x^k D_d u\in L^p(H;dv)$, $0\leqq|k|\leqq s-1$ with $s=[(d-1)/p]+1$. Then u belongs to*

$C^{\lambda}(\overline{H})$, *where* $0<\lambda\leqq s-(d-1)/p$ *or* $0<\lambda<1$ *according to* $(s-1)p<d-1<sp$ *or* $d-1=(s-1)p$.

Proof. Since $D_d^{\varepsilon}\mathbf{D}_{x'}^{k+l}u=D_{x'}^{k+l}D_d^{\varepsilon}u=D_{x'}^{k}D_d^{\varepsilon}D_{x'}^{l}u$ a.e. $(0\leqq|k|+|l|\leqq s,\ \varepsilon=0,1)$ and $\tilde{D}_dD_{x'}^{k}D_du=D_{x'}^{k}\tilde{D}_dD_du$ ν-a.e. $(0\leqq|k|\leqq s-1)$, it is enough to prove that there is a version v which satisfies $u=v$ ν-a.e. and

$$\sup_{\substack{x,\,y\in H\\ x\neq y}}|v(x)-v(y)|/|x-y|^{\lambda}\leqq C_3(d,p,h)\Big\{\sum_{\substack{0\leqq|k|\leqq s\\ \varepsilon=0,1}}\|D_{x'}^{k}D_d^{\varepsilon}u;L^p(H;dx)\| + \sum_{0\leqq|k|\leqq s-1}\|\mathbf{D}_{x'}^{k}\tilde{D}_dD_du;L^p(H;d\nu)\|\Big\}. \tag{16}$$

Let $x\in H$ and let $H(\sigma)=H(b,\sigma/2)$ be any cube such that $x\in\overline{H(\sigma)}\subset H$. Put $H_d=(a_d-h,a_d+h)$, $H'=\{x'=(x_1,\dots,x_{d-1})\in R^{d-1}\colon (x_1,\dots,x_d)\in H$ for some $x_d\in R^1\}$ etc. Since u, together with $\mathbf{D}_iu$, $1\leqq i\leqq d-1$, belongs to $L^p(H;d\nu)$, $u(\cdot,x_d)$ can be approximated in $W^{1,p}(H')$ (=Sobolev space) by functions smooth on H' for m-a.e. $x_d\in H_d$. Therefore

$$\left|u(x',x_d)-\frac{1}{\sigma^{d-1}}\int_{H(\sigma)'}u(z',x_d)\,dz'\right|\leqq\frac{1}{\sigma^{d-1}}\int_{\Sigma}\sum_{i=1}^{d-1}|D_iu(x'+r\xi,x_d)|\rho^{d-2}\,d\rho\,d\xi\,dr,$$

where we used polar coordinates for z' with center $x'\colon z'=x'+\rho\xi$, $\rho=|x'-z'|$, $\xi\in S^{d-2}$ (=unit sphere in R^{d-1}) and $\Sigma=\{(\rho,\xi,r)\colon x'+\rho\xi\in H(\sigma)',\ 0\leqq r\leqq\rho\}$. Then by $\Sigma\subset\{(\rho,\xi,r)\colon 0\leqq\rho\leqq\sqrt{d-1}\,\sigma,\ x'+r\xi\in H(\sigma)'\}$ and by Proposition 1−(ii)

$$\leqq c_1(d)\int_{H(\sigma)'}\sum_{i=1}^{d-1}\left(\frac{1}{2h}\int_{H_d}|D_iu(y',t)|\,dt+\int_{H_d}|D_dD_iu(y',t)|\,dt\right)\frac{dy'}{|x'-y'|^{d-2}}$$

for ν-a.e. $x\in H$. Set

$$\begin{aligned} &r=(d-1)p/\{(d-1)-(s-1)p\} && \text{if } (s-1)p<d-1<sp,\\ &p<r<\infty,\ 0<1-(d-1)/r(<1) && \text{if } d-1=(s-1)p. \end{aligned}$$

Hölder's inequality and the Sobolev imbedding theorem give

$$\begin{aligned} &\left|u(x',x_d)-\frac{1}{\sigma^{d-1}}\int_{H(\sigma)'}u(z',x_d)\,dz'\right|\\ &\leqq c_2(d,r,h)\sigma^{1-(d-1)/r}\sum_{\substack{\varepsilon=0,1\\ |l|=1}}\int_{H_d}\Big(\int_{H(\sigma)'}|D_d^{\varepsilon}D_{x'}^{l}u(y',t)|^r\,dy'\Big)^{1/r}dt\\ &\leqq c_3(d,s,p,r,h)\sigma^{1-(d-1)/r}\sum_{\substack{0\leqq|k|\leqq s-1\\ \varepsilon=0,1,\,|l|=1}}\int_{H_d}\Big(\int_{H'}|D_{x'}^{k}D_d^{\varepsilon}D_{x'}^{l}u(y',t)|^p\,dy'\Big)^{1/p}dt\\ &\leqq c_4(d,s,p,r,h)\sigma^{1-(d-1)/r}\sum_{\substack{0\leqq|k|\leqq s\\ \varepsilon=0,1}}\|D_{x'}^{k}D_d^{\varepsilon}u;L^p(H;dx)\|,\ \nu\text{-a.e. } x. \end{aligned}$$

On the other hand, we have a version w such that $w=D_du$ a.e., $\tilde{D}_dw=\tilde{D}_dD_du$ ν-a.e. and for $\hat{\nu}_d$-a.e. x'

$$w(x',x_d)-w(x',t)=\begin{cases}\displaystyle\int_{[t,\,x_d)}\tilde{D}_dw(x',s)m(ds) & \text{if } t<x_d,\\ \displaystyle-\int_{[x_d,\,t)}\tilde{D}_dw(x',s)m(ds) & \text{if } t\geqq x_d.\end{cases}$$

Therefore for the same reason as above we get

$$\begin{aligned}
&\frac{1}{\nu(H(\sigma))}\int_{H(\sigma)}|u(z',x_d)-u(z',z_d)|\,\nu(dz)\\
&\leqq\frac{1}{\nu(H(\sigma))}\int_{H(\sigma)}\nu(dz)\int_{H_d}|D_d u(z',t)|\,dt\\
&\leqq\frac{1}{\sigma^{d-2}}\int_{H(\sigma)'}\left(\frac{1}{2h}\int_{H_d}|D_d u(z',t)|\,dt+\int_{H_d}|\tilde{D}_d D_d u(z',t)|m(dt)\right)dz'\\
&\leqq\sigma^{1-(d-1)/r}\left\{\frac{1}{2h}\int_{H_d}\Big(\int_{H(\sigma)'}|D_d u(z',t)|^r dz'\Big)^{1/r}dt\right.\\
&\quad\left.+\int_{H_d}\Big(\int_{H(\sigma)'}|\tilde{D}_d D_d u(z',t)|^r dz'\Big)^{1/r}m(dt)\right\}\\
&\leqq c_5(d,s,p,r,h)\sigma^{1-(d-1)/r}\sum_{0\leqq|k|\leqq s-1}\Big\{\int_{H_d}\Big(\int_{H'}|D^k_{x'}D_d u(z',t)|^p dz'\Big)^{1/p}dt\\
&\quad+\int_{H_d}\Big(\int_{H'}|D^k_{x'}\tilde{D}_d D_d u(z',t)|^p dz'\Big)^{1/p}m(dt)\Big\}\\
&\leqq c_6(d,s,p,r,h)\sigma^{1-(d-1)/r}\sum_{0\leqq|k|\leqq s-1}\big(\|D^k_{x'}D_d u;L^p(H;dx)\|\\
&\quad+\|D^k_{x'}\tilde{D}_d D_d u;L^p(H;dv)\|\big),
\end{aligned}$$

for m-a.e. x_d. Hence

$$\begin{aligned}
&|u(x',x_d)-\frac{1}{\nu(H(\sigma))}\int_{H(\sigma)}u(z',z_d)\,\nu(dz)|\\
&\leqq c_7(d,s,p,r,h)\sigma^{1-(d-1)/r}\Big(\sum_{\substack{0\leqq|k|\leqq s\\ \varepsilon=0,1}}\|D^k_{x'}D^\varepsilon_d u;L^p(H;dx)\|\\
&\quad+\sum_{0\leqq|k|\leqq s-1}\|\mathbf{D}^k_{x'}\tilde{D}_d D_d u;L^p(H;dv)\|\Big),\qquad\text{for }\nu\text{-a.e. }x.
\end{aligned}$$

For ν-a.e. $x, y\in H$ having $\sigma=|x-y|$ we can take a cube $H(\sigma)=H(b,\sigma/2)$ such that $x, y\in\overline{H(\sigma)}\subset H$. Thus (16) follows. q.e.d.

Dual Space. We shall turn our attention to the dual space of $\mathscr{F}^p_0(\mathrm{R}^d;\nu)$, which is denoted by $\mathscr{F}^{-p'}(R^d;\nu)$ or simply $\mathscr{F}^{-p'}$ with $p'=p/(p-1)$. Let $\mathscr{A}^p$ and $\mathscr{B}^p$ be linear subspaces of distributions such that: $T_1\in\mathscr{A}^p$ and $T_2\in\mathscr{B}^p$ are expressed as

$$\langle T_1,\varphi\rangle=\int_{R^d}f_1\varphi\,d\nu+\int_{R^d}f_2\varphi\,dx,\tag{17}$$

$$\langle T_2,\varphi\rangle=\int_{R^d}(g_1\varphi+g_2 D_d\varphi)dx,\tag{18}$$

for every $\varphi\in C_0^\infty(R^d)$ and for some $f_1\in L^2(R^d;d\nu)\cap L^p(R^d;d\nu)$ and f_2, g_1, $g_2\in L^2(R^d;dx)\cap L^p(R^d;dx)$. Obviously $\mathscr{A}^p\cup\mathscr{B}^p\subset\mathscr{F}^{-2}\cap\mathscr{F}^{-p}$.

Proposition 4. (*i*) *If an element* $T\in\mathscr{A}^p$ *is given as* (17) *and* $\mathrm{Supp}[f_1]$ *is compact, then* T *belongs to* $\mathscr{B}^p$. (*ii*) *For any* $T\in\mathscr{B}^p$ *and any* $\varepsilon>0$ *there are* $\Phi\in C_0^\infty(R^d)$ *and*

$h_i \in L^2(R^d; dx) \cap L^p(R^d; dx)$, $i=1, 2$ *such that* $\|h_i\|_2 < \varepsilon$, $\|h_i\|_p < \varepsilon$ *and*

$$\langle T, \varphi \rangle - \int_{R^d} \Phi \varphi dx = \int_{R^d} (h_1 \varphi + h_2 D_d \varphi) dx \qquad \varphi \in \mathscr{F}_0^2 \cap \mathscr{F}_0^{p'}, \tag{19}$$

where $\|\cdot\|_p$ *stands for the norm of* $L^p(R^d; dx)$. *(iii) Suppose that an element* $T \in \mathscr{A}^p$ *is expressed as* (17) *in terms of functions* f_i, $i=1, 2$ *such that both* Supp$[f_i]$*'s are contained in a closed ball* Σ. *Then for given open ball* B *with* $B \cap \Sigma = \phi$ *a function* Φ *in (ii) can be chosen to be* $\Phi = 0$ *on* B.

Proof. Assume the condition of (i). Let $H(h) = H(0, h)$ be a cube containing Supp$[f_1]$ and put

$$F(x', x_d) = \begin{cases} \int_{(x_d, 2h)} f_1(x', t) m(dt) & \text{if } (x', x_d) \in H(2h), \\ 0 & \text{otherwise.} \end{cases}$$

We choose an $\alpha \in C_0^\infty((-2h, 2h))$ satisfying $\alpha = 1$ on $[-h, h]$, and find

$$\int f_1 \varphi dv = \int F D_d(\alpha \varphi) dx \qquad \varphi \in C_0^\infty(R^d).$$

Since F belongs to $L^2(R^d; dx) \cap L^p(R^d; dx)$ we get the assertion of (i).

Part (ii) is obvious. In fact, for any $T \in \mathscr{B}^p$ and for any $\varepsilon > 0$ there are functions $\psi_i \in C_0^\infty(R^d)$, $i=1, 2$ such that $\|g_i - \psi_i\|_2 < \varepsilon$ and $\|g_i - \psi_i\|_p < \varepsilon$, where g_i, $i=1, 2$ are functions expressing T. Setting $\Phi = \psi_1 - D_d \psi_2$, $h_1 = g_1 - \psi_1$, $h_2 = g_2 - \psi_2$ leads us to (19).

Let us show the statement (iii). By $b = (b_1, \ldots, b_d)$ and $\sigma = (\sigma_1, \ldots, \sigma_d)$ denote the centers of B and Σ respectively. Selecting a cube $H(h)$ such that $\Sigma \subset H(h)$, we define the function

$$G(x', x_d) = \begin{cases} \int_{(x_d, 2h)} f_1(x', t) m(dt) & \text{if } (x', x_d) \in H(2h), \sigma_d \leqq b_d, \\ -\int_{(-2h, x_d)} f_1(x', t) m(dt) & \text{if } (x', x_d) \in H(2h), \sigma_d > b_d, \\ 0 & \text{otherwise.} \end{cases}$$

Repeat the above arguments replacing F, g_1 and g_2 by $G, f_2 + \alpha' G$ and αG respectively. Since $G = 0$ on B, we obtain conclusion. q.e.d.

2. Diffusion Processes on R^d

Let m be the same measure as in the previous section, that is, m is a nonnegative measure on R^1 satisfying (6). ν_{ij}'s are defined by (2). a_{ij}, $i, j=1, \ldots, d$ satisfy the following:

(20) For each i, j $a_{ji} = a_{ji}$ and a_{ij} is of C^{d_0}-class on R^d.

(21) There is a constant $\gamma(\geqq 1)$ such that

$$\gamma^{-1} \sum_{i=1}^d \xi_i^2 \nu_{ii}(dx) \leqq \sum_{i,j=1}^d \xi_i \xi_j a_{ij}(x) \nu_{ij}(dx) \leqq \gamma \sum_{i=1}^d \xi_i^2 \nu_{ii}(dx) \qquad x, \xi \in R^d.$$

Abbreviating $\mathscr{F}_0^2(R^d; \nu)$ as $\mathscr{F}$, we now define the form $\mathscr{B}$ by

$$\mathscr{D}[\mathscr{B}] = \mathscr{F} \tag{22}$$
$$\mathscr{B}(u, v) = \sum_{i,j=1}^{d} \int_{R^d} D_i u(x) D_j v(x) a_{ij}(x) \nu_{ij}(dx),$$

where derivatives $D_i u, 1 \leqq i \leqq d-1$ are interpreted in the weak sense. For $\lambda > 0$ put $\mathscr{B}_\lambda(\ ,\ ) = \mathscr{B}(\ ,\ ) + \lambda(\ ,\ )$, $(\ ,\ )$ being the inner product of $L^2(R^d; dx)$. It follows from (10), (20) and (21) that $\mathscr{B}$ is a Dirichlet form on $L^2(R^d; dx)$. Namely, (i) $\mathscr{F}$ is a dense subspace of $L^2(R^d; dx)$, (ii) $\mathscr{F}$ is a Hilbert space equipped with the inner product $\mathscr{B}_1(\ ,\ )$, (iii) $\mathscr{B}$ is symmetric and nonnegative definite, and (iv) if $u \in \mathscr{F}$, then $v \equiv (0 \vee u) \wedge 1 \in \mathscr{F}$ and $\mathscr{B}(v, v) \leqq \mathscr{B}(u, u)$, where $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Particularly, in this case we notice that: Let α be a function on R^1 having the properties $\alpha(0) = 0$ and $0 \leqq \alpha(t) - \alpha(s) \leqq t - s$ if $s \leqq t$. If $u \in \mathscr{F}$, then $\alpha(u) \in \mathscr{F}$ and

$$\mathscr{B}(\alpha(u), \alpha(u)) \leqq \mathscr{B}(\alpha(u), u) \leqq \mathscr{B}(u, u). \tag{23}$$

Let T be an element of $\mathscr{F}^{-2}$ and u be a solution of the equation

$$u \in \mathscr{F} \tag{24}$$
$$\mathscr{B}_\lambda(u, \varphi) = \langle T, \varphi \rangle \quad \varphi \in \mathscr{F}.$$

For each $\lambda > 0$ we define the operator G_λ from $\mathscr{F}^{-2}$ into $\mathscr{F}$ by setting $u = G_\lambda T$. The Riesz representation theorem asserts that $G_\lambda T$ is determined uniquely by $\lambda > 0$ and $T \in \mathscr{F}^{-2}$. $\{G_\lambda : \lambda > 0\}$ is called the resolvent associated with $\mathscr{B}$. It is evident by the definition that $\{G_\lambda : \lambda > 0\}$ satisfies the resolvent equation:

$$G_\lambda - G_\eta + (\lambda - \eta) G_\lambda G_\eta = 0 \quad \lambda, \eta > 0. \tag{25}$$

With the aid of results due to J. Deny [3], we see that λG_λ is subMarkov for each $\lambda > 0$:

$$0 \leqq \lambda G_\lambda f \leqq 1 \quad \text{whenever} \quad 0 \leqq f \leqq 1 \quad \text{and} \quad f \in L^2(R^d; dx). \tag{26}$$

Moreover the resolvent fulfils the following:

Proposition 5. *$G_\lambda(C_0)$ is a dense subset of C_∞ for each $\lambda > 0$.*

For the proof we need auxiliary lemmas. Noting (10), (12), (21) and (23), we can easily obtain the following lemma in the same way as [14; Theorem 4.2] (cf. [16; Theorem 4.2]).

Lemma 6. *If $p > d$ and $T \in \mathscr{B}^p$, then*

$$\| G_\lambda T \|_\infty \leqq C_4(\gamma, \lambda, d, p, A, M) \sum_{i=1,2} (\| g_i \|_2 + \| g_i \|_p), \tag{27}$$

where $g_i, i = 1, 2$ are expressing T as (18).

Next we turn to the Hölder continuity of $G_\lambda T$. Following L. Nirenberg's argument [13], we have already gotten in [16] the result corresponding to $m_1 = \delta$-measure on a bounded cube. In the present case we can still utilize his idea.

Lemma 7. *Let $T\in\mathscr{A}^2$ be represented as* (17) *by means of $f_i, i=1,2$ such that $\mathbf{D}^k_{x'}f_1\in L^2(R^d;dv)$ and $D^k_{x'}f_2\in L^2(R^d;dx)$ for $0\leqq|k|\leqq d_0-1$. Then $G_\lambda T$ is Hölder continuous on R^d with exponent l: $0<l\leqq 1/2$ if d is even or $0<l<1$ if d is odd.*

Proof. Let $H=H(a,1)$. Set $u=G_\lambda T$. For any $r(1\leqq r\leqq d-1)$ and $\varphi\in\mathscr{F}$ we denote by $\Delta_h\varphi$ the difference quotient: $\Delta_h\varphi(x)=(\varphi(x+he_r)-\varphi(x))/h$, where $h\in R^1$ and e_r is a unit vector along the x_r-axis. $\mathscr{B}_\lambda(u,\alpha\Delta_h\varphi)=\langle T,\alpha\Delta_h\varphi\rangle$ for every $\varphi\in\mathscr{F}$ and $\alpha\in C_0^\infty(H)$, whence

$$\mathscr{B}_\lambda(\Delta_{-h}(\alpha u),\varphi)=-\langle T,\alpha\Delta_h\varphi\rangle+\sum_{i,j=1}^{d}\int_H\{(D_iu)(D_j\alpha)(\Delta_h\varphi)a_{ij} \tag{28}$$
$$-(D_i\alpha)u(D_j\Delta_h\varphi)a_{ij}-D_i(\alpha u)(D_j\varphi)(\cdot+he_r)\Delta_h a_{ij}\}\,dv_{ij},$$

where D_i means $\mathbf{D}_i$ for $1\leqq i\leqq d-1$. On the other hand, in the similar way to Sobolev spaces one can easily show

$$\int_{R^d}(\Delta_h\varphi)^2\,dv\leqq\int_{R^d}(\mathbf{D}_r\varphi)^2\,dv,\quad\text{and} \tag{29}$$

$$\lim_{h\to 0}\int_{R^d}(\Delta_h\varphi-\mathbf{D}_r\varphi)^2\,dv=0. \tag{30}$$

Combining (29) with (20), we derive from (28) that

$$|\mathscr{B}_\lambda(\Delta_{-h}(\alpha u),\varphi)|\leqq c_1|\varphi|_{2,H,\nu}\qquad\varphi\in\mathscr{F}_0^2(H;\nu),$$

c_1 being independent of φ and h. Taking $\varphi=\Delta_{-h}(\alpha u)(\in\mathscr{F}_0^2(H;\nu))$ with sufficiently small h, we find by (21)

$$(\gamma^{-1}\wedge\lambda)|\Delta_{-h}(\alpha u)|^2_{2,H,\nu}\leqq\mathscr{B}_\lambda(\Delta_{-h}(\alpha u),\Delta_{-h}(\alpha u))\leqq c_1|\Delta_{-h}(\alpha u)|_{2,H,\nu}$$

from which $\{\Delta_{-h}(\alpha u)\}$ is a bounded set of $\mathscr{F}_0^2(H;\nu)$. In cosideration of (30) we apply the Banach - Saks theorem to obtain a subsequence $\{\Delta_{-h_n}(\alpha u)\}$ whose Cesáro sums converge to $\mathbf{D}_r(\alpha u)$ in $\mathscr{F}_0^2(H;\nu)$. Repeating this argument we have

$$\mathbf{D}^k_{x'}(\alpha u)\in\mathscr{F}_0^2(H;\nu)\qquad 0\leqq|k|\leqq d_0. \tag{31}$$

Now $\mathscr{B}_\lambda(u,\alpha D^k_{x'}\varphi)=\langle T,\alpha D^k_{x'}\varphi\rangle$ for every $\varphi\in C_0^\infty(H)$ and $\alpha\in C_0^\infty(H)$. Since $D^\varepsilon_d\mathbf{D}^k_{x'}u=D^k_{x'}D^\varepsilon_d u$ a.e. $(\varepsilon=0,1)$, an integration by parts gives

$$\int D^k_{x'}(a_{dd}D_d(\alpha u))D_d\varphi\,dx=\int\mathbf{D}^k_{x'}(\alpha f_1)\varphi\,dv+\int D^k_{x'}(\alpha f_2)\varphi\,dx-\lambda\int D^k_{x'}(\alpha u)\varphi\,dx$$
$$+\sum_{i,j=1}^{d-1}\int\mathbf{D}^k_{x'}(\alpha\mathbf{D}_j(a_{ij}\mathbf{D}_iu))\varphi\,dv+\sum_{j=1}^{d-1}\int D^k_{x'}(\alpha D_j(a_{dj}D_d u))\varphi\,dx$$
$$-\int D^k_{x'}D_d(a_{dd}uD_d\alpha)\varphi\,dx-\int D^k_{x'}(a_{dd}D_duD_d\alpha)\varphi\,dx,\qquad 0\leqq|k|\leqq d_0-1.$$

Therefore we get by Riesz's representation theorem

$$\int_H D^k_{x'}(a_{dd}D_d(\alpha u))D_d\varphi\,dx=\int_H F\varphi\,dv$$

for some $F \in L^2(H; dv)$ and any $\varphi \in C_0^\infty(H)$, which implies $\tilde{D}_d D_{x^k} \cdot (a_{dd} D_d(\alpha u)) \in L^2(H; dv)$, and so $\tilde{D}_d D_{x^k} \cdot D_d(\alpha u) \in L^2(H; dv), 0 \leqq |k| \leqq d_0 - 1$ because a_{dd} is positive and smooth.

We have just shown that αu satisfies the conditions of Proposition 3. Hence $\alpha u \in C^1(\overline{H})$. Since α and H are arbitrary, we thus obtain the conclusion. q.e.d.

We are finally in a position to show

Lemma 8. *$G_\lambda(\mathscr{B}^p) \subset C_\infty$ for each $p > d$ and $\lambda > 0$.*

Proof. Let T be any element of $\mathscr{B}^p$. By Proposition 4-(ii) and Lemmas 6 and 7 we have a sequence $\{\Phi_n\} \subset C_0^\infty(R^d)$ such that $G_\lambda \Phi_n$ converges uniformly on R^d to $G_\lambda T$. To complete the proof it suffices to show $G_\lambda(C_0^\infty(R^d)) \subset C_\infty$. Let us fix $\lambda > 0$ and $\Phi \in C_0^\infty(R^d)$ and put $u = G_\lambda \Phi$. Choose a unit cube $H \equiv H(a, 1/2)$ such that $H \cap \mathrm{Supp}[\Phi] = \phi$. Then $\mathscr{B}_\lambda(u, \varphi) = 0$ for all $\varphi \in C_0^\infty(H)$. By using (11) and (21) we can repeat the arguments in [14; Theorem 5.1] (cf. [16; Theorem 4.3]) to get $\sup\{|u(x)|: x \in H(a, 1/4)\} \leqq c(\gamma, d, A) \| u;\ L^2(H; dv) \|$. Thus we obtain the conclusion in the same way as [16; Proposition 5.1]. q.e.d.

We now proceed to a

Proof of Proposition 5. We use an idea of Kunita [12]. Fix any p with $p > d$. Let us define the operator L from $C_0^\infty(R^d)$ into $\mathscr{A}^p \cap \mathscr{B}^p$ as follows: For each $u \in C_0^\infty(R^d)$

$$\langle Lu, \varphi \rangle = \sum_{i,j=1}^{d} \int_{R^d} D_i(a_{ij} D_j u)\, \varphi \, dv_{ij} \qquad \varphi \in C_0^\infty(R^d). \tag{32}$$

What Lu belongs to $\mathscr{A}^p \cap \mathscr{B}^p$ is derived from the fact: $D_i(a_{ij} D_j u) \in C_0$, $1 \leqq i, j \leqq d$ and Proposition 4-(i). Furthermore we set $T = (\lambda - L) u (\in \mathscr{A}^p \cap \mathscr{B}^p)$, where $\lambda (> 0)$ is arbitrarily fixed. Then by Lemma 8 $G_\lambda(\mathscr{B}^p)$ is a dense subset of C_∞. An application of Proposition 4-(ii) and Lemma 6 finishes the proof. q.e.d.

By (25), (26) and Proposition 5 we can appeal to the Hille-Yoshida theorem to get a nonnegative, strongly continuous and subMarkov semigroup $\{T_t: t > 0\}$ on C_∞ such that

$$G_\lambda f(x) = \int_0^\infty e^{-\lambda t} T_t f(x)\, dt \qquad f \in C_\infty, \lambda > 0.$$

Therefore we are also given a unique associated Hunt process $X = [x(t), \zeta, P_x]$ on R^d.

Let us show the continuity of the sample paths following Kanda's arguments [11]. Let B be an arbitrary open ball and Ω be another open ball with $\Omega \cap B = \phi$. Then there is a function $u \in C_0^\infty(R^d)$ satisfying $u > 0$ in Ω and $u = 0$ outside Ω. Considering Lu defined by (32) with such u, we set $T = (\lambda - L)u$, where $\lambda > 0$ is fixed arbitrarily. Notice $u = G_\lambda T$. Since T fulfils the conditions in Proposition 4-(iii) replaced by $\overline{\Omega}$ instead of Σ, by Lemma 6 we have functions Φ_n, $n = 1, 2, \ldots \in C_0^\infty(R^d)$ such that their supports are contained in B^c and $G_\lambda \Phi_n$ converges uniformly to u. Put $\tau = \inf\{t > 0: x_t \notin B\}$. Then by Dynkin's formula $G_\lambda \Phi_n(x) = E_x[e^{-\lambda \tau} G_\lambda \Phi_n(x_\tau)]$ $(x \in B)$. Making $n \to \infty$, we get $u(x) = E_x[e^{-\lambda \tau} u(x_\tau)] = 0 (x \in B)$, that is, $P_x(x_\tau \in \Omega) = 0 (x \in B)$. Since Ω is arbitrary, we are led to $P_x(x_\tau \in \partial B) = 1$, which means the continuity of sample paths of X.

Finally we arrive at

Theorem 9. *For the form $\mathscr{B}$ defined by* (22) *there exists a unique diffusion process* $X=[x(t), \zeta, P_x]$ *on* R^d *such that*

$$G_\lambda f(x)=E_x[\int_0^\zeta e^{-\lambda t} f(x_t)\,dt] \quad f\in C_\infty, \lambda>0.$$

3. Proof of the Theorem

Closability of $\mathscr{Q}$. Let $\{u_n\}\subset C_0^\infty(\Omega)$ be a sequence such that $\mathscr{Q}(u_m-u_n, u_m-u_n)\to 0$ as $m, n\to\infty$ and $(u_n, u_n)_\Omega\to 0$ as $n\to\infty$. It is sufficient to verify that $\mathscr{Q}(u_n, u_n)\to 0$ as $n\to\infty$. First we show that the

$$\lim_{n\to\infty}\sum_{i=1}^d\int_\Omega (D_i(\alpha u_n))^2\,dv_{ii}=0 \quad \alpha\in C_0^\infty(\Omega). \tag{33}$$

For any $\alpha\in C_0^\infty(\Omega)$ there is a closed cube $H=\{x\in R^d: l_i\leqq x_i\leqq r_i,\ 1\leqq i\leqq d\}$ such that $\mathrm{Supp}[\alpha]\subset H$, $\inf\{m_0(t): l_d\leqq t\leqq r_d\}>0$ and $m([l_d, r_d])<\infty$. Then in the same way as (9)

$$\int_\Omega(\alpha u_m-\alpha u_n)^2\,dv\leqq c_1\{\int_H(\alpha u_m-\alpha u_n)^2\,dx+\int_H(D_d(\alpha u_m-\alpha u_n))^2\,dx\} \tag{34}$$

for some c_1, and by (3)

$$\leqq c_2\{(u_m-u_n, u_m-u_n)_\Omega+\mathscr{Q}(u_m-u_n, u_m-u_n)\}\to 0 \text{ as } m, n\to\infty,$$

for an appropriate c_2. Therefore by using (3) again, we have

$$\begin{aligned}&\lim_{m,n\to\infty}\sum_{i=1}^d\int_\Omega(D_i(\alpha u_m-\alpha u_n))^2\,dv_{ii}\leqq 2\lim_{m,n\to\infty}\sum_{i=1}^d\int_\Omega(\alpha D_i(u_m-u_n))^2\,dv_{ii}\\ &\leqq c_3\lim_{m,n\to\infty}\mathscr{Q}(u_m-u_n, u_m-u_n)=0,\end{aligned} \tag{35}$$

with some c_3 and for every $\alpha\in C_0^\infty(\Omega)$. From now on α is fixed arbitrarily. Since $\{D_i(\alpha u_n)\}$ is Cauchy sequence in $L^2(\Omega; dx)$ for each i, and since $(\alpha u_n, \alpha u_n)_\Omega\to 0$ as $n\to\infty$, the closability of Dirichlet integrals ensures

$$\lim_{n\to\infty}\int_\Omega(D_d(\alpha u_n))^2\,dx=0. \tag{36}$$

Hence by the same method as (34), we obtain

$$\begin{aligned}&\left|\sum_{i=1}^{d-1}\int_\Omega D_i(\alpha u_n)D_i(\alpha u_m)\,dv\right|=\left|\sum_{i=1}^{d-1}\int_\Omega D_i^2(\alpha u_n)\alpha u_m\,dv\right|\\ &\leqq c_4\left\{\sum_{i=1}^{d-1}\int_\Omega(D_i^2(\alpha u_n))^2\,dv\right\}^{1/2}\left\{\int_\Omega(\alpha u_m)^2\,dx+\int_\Omega(D_d(\alpha u_m))^2\,dx\right\}^{1/2}\\ &\to 0 \quad \text{as } m\to\infty,\end{aligned}$$

for each n and for some c_4. Combining this with (35) and (36), we get (33). Indeed,

$$\lim_{n\to\infty}\sum_{i=1}^{d}\int_{\Omega}(D_i(\alpha u_n))^2\,dv_{ii}=\lim_{n\to\infty}\sum_{i=1}^{d-1}\int_{\Omega}(D_i(\alpha u_n))^2\,dv$$
$$\leqq\lim_{n\to\infty}\limsup_{m\to\infty}\left\{\sum_{i=1}^{d-1}\int_{\Omega}(D_i(\alpha u_n-\alpha u_m))^2\,dv+2\left|\sum_{i=1}^{d-1}\int_{\Omega}D_i(\alpha u_n)\,D_i(\alpha u_m)\,dv\right|\right\}$$
$$=0.$$

Put $\Omega_1=\{x\in\Omega\colon |x|<1,\ \mathrm{dist}(x,\partial\Omega)>1\}$ and $\Omega_l=\{x\in\Omega\colon |x|<l,\ \mathrm{dist}(x,\partial\Omega)>1/l\}-\Omega_{l-1}(l\geqq 2)$. Then by using (33) we find

$$0\leqq\mathscr{Q}(u_n,u_n)=\sum_{l=1}^{\infty}\sum_{i,j=1}^{d}\int_{\Omega_l}D_iu_nD_ju_n\,a_{ij}\,dv_{ij}$$
$$\leqq\sum_{l=1}^{\infty}\liminf_{m\to\infty}\left\{\sum_{i,j=1}^{d}\int_{\Omega_l}D_i(u_n-u_m)\,D_j(u_n-u_m)\,a_{ij}\,dv_{ij}\right.$$
$$\left.+2\sum_{i,j=1}^{d}\int_{\Omega_l}D_iu_nD_ju_m\,a_{ij}\,dv_{ij}\right\}$$
$$\leqq\sum_{l=1}^{\infty}\liminf_{m\to\infty}\left[\sum_{i,j=1}^{d}\int_{\Omega_l}D_i(u_n-u_m)\,D_j(u_n-u_m)\,a_{ij}\,dv_{ij}\right.$$
$$\left.+2\gamma(\Omega_l)\left\{\sum_{i=1}^{d}\int_{\Omega_l}(D_iu_n)^2\,dv_{ii}\right\}^{1/2}\left\{\sum_{i=1}^{d}\int_{\Omega_l}(D_iu_m)^2\,dv_{ii}\right\}^{1/2}\right]$$
$$\leqq\liminf_{m\to\infty}\sum_{l=1}^{\infty}\sum_{i,j=1}^{d}\int_{\Omega_l}D_i(u_n-u_m)\,D_j(u_n-u_m)\,a_{ij}\,dv_{ij}$$
$$=\liminf_{m\to\infty}\mathscr{Q}(u_n-u_m,u_n-u_m),$$

whence $\lim_{n\to\infty}\mathscr{Q}(u_n,u_n)=0$.

A Diffusion Process on Ω. We divide the proof into two steps.

Step 1. Assume that we have a domain Ω^1 such that $\overline{\Omega}\subset\Omega^1$, m is defined on Ω_d^1, $\inf\{m_0(t)\colon t\in\Omega_d^1\}>0$ and $m(\Omega_d^1)<\infty$. Moreover every a_{ij} is of C^{d_0}-class on Ω^1 and there is a constant $\gamma(\geqq 1)$ such that

$$\gamma^{-1}\sum_{i=1}^{d}\xi_i^2\,\nu_{ii}(dx)\leqq\sum_{i,j=1}^{d}\xi_i\xi_j\,a_{ij}(x)\,\nu_{ij}(dx)\leqq\gamma\sum_{i=1}^{d}\xi_i^2\,\nu_{ii}(dx)\qquad x\in\Omega^1,\ \xi\in R^d.$$

Under these assumptions the form $\mathscr{E}$, the smallest closed extension of $\mathscr{Q}$, is given as

$$\mathscr{D}[\mathscr{E}]=\mathscr{F}_0^2(\Omega;\nu)$$
$$\mathscr{E}(u,v)=\sum_{i,j=1}^{d}\int_{\Omega}D_iuD_jv\,a_{ij}\,dv_{ij}$$

Here derivatives D_i, $1\leqq i\leqq d-1$ are taken in the weak sense.

We shall extend this form as follows: Put $\tilde{m}=m$ on Ω_d^1, or $=$the one-dimensional Lebesgue measure outside Ω_d^1 and set $\tilde{\nu}(dx_1 \dots dx_d) = dx_1 \dots dx_{d-1}\tilde{m}(dx_d)$. Define $\tilde{\nu}_{ij}$, $1 \leqq i,j \leqq d$ on R^d by (2) replaced by $\tilde{\nu}$ instead of ν. By the above assumptions we have also $\tilde{a}_{ij}$, $1 \leqq i,j \leqq d$ on R^d such that $\tilde{a}_{ij}=a_{ij}$ on Ω and they possess the properties (20) and (21) with $\tilde{a}_{ij}$ and $\tilde{\nu}_{ij}$ in place of a_{ij} and ν_{ij} respectively (see [16; Lemma 5.6]). Let us introduce the form $\tilde{\mathscr{E}}$:

$$\mathscr{D}[\tilde{\mathscr{E}}] = \mathscr{F}_0^2(R^d; \tilde{\nu})$$

$$\tilde{\mathscr{E}}(u,v) = \sum_{i,j=1}^{d} \int_{R^d} D_i u D_j v \, \tilde{a}_{ij} d\tilde{\nu}_{ij}$$

where $D_i = \mathbf{D}_i$ for $1 \leqq i \leqq d-1$. This may be considered an extension of $\mathscr{E}$.

Since $\tilde{\mathscr{E}}$ fulfills the conditions in the previous section, Theorem 9 asserts that there exists a unique diffusion process $\tilde{X}=[\tilde{x}(t), \zeta, \tilde{P}_x]$ $(x \in R^d)$ whose resolvent $\tilde{R}_\lambda$ defines a operator on $L^2(R^d; dx)$ and satisfies the equation

$$\tilde{R}_\lambda f \in \mathscr{D}[\tilde{\mathscr{E}}]$$

$$\tilde{\mathscr{E}}_\lambda(\tilde{R}_\lambda f, \varphi) = (f, \varphi) \qquad \varphi \in \mathscr{D}[\tilde{\mathscr{E}}]$$

for every $f \in L^2(R^d; dx)$ and $\lambda > 0$.

Let $\tilde{\tau} = \inf\{t > 0 : \tilde{x}(t) \notin \Omega\}$ and set $\tilde{X}_\Omega = [\tilde{x}(t \wedge \tilde{\tau}), \tilde{\tau}, \tilde{P}_x]$ $(x \in \Omega)$, which is called a part of $\tilde{X}$ on Ω. With the aid of a remarkable result due to M. Fikushima [6; Theorem 4.4.2] we find that if $\{\tilde{R}_{\Omega\lambda} : \lambda > 0\}$ denotes the resolvent operator of $\tilde{X}_\Omega$, then $\tilde{R}_{\Omega\lambda} f$ is a solution of

$$(37) \qquad \begin{aligned} &u \in \mathscr{F}_0^2(\Omega; \nu) \\ &\tilde{\mathscr{E}}_\lambda(u, \varphi) = (f, \varphi)_\Omega \qquad \varphi \in \mathscr{F}_0^2(\Omega; \tilde{\nu}) \end{aligned}$$

where $f \in L^2(\Omega; dx)$ and $\lambda > 0$. Since $\tilde{\nu} = \nu$ on Ω, $\tilde{\mathscr{E}} = \mathscr{E}$ on $\mathscr{F}_0^2(\Omega; \nu)$ and a solution of (37) is unique according to the Riesz representation theorem, $\tilde{X}_\Omega$ is none other than the desired process.

Step 2. We turn to the general situation of the theorem. Let $\{\Omega_n\}$ be an increasing sequence of bounded subdomains of Ω such that $\bigcup_n \Omega_n = \Omega$ and each closure $\overline{\Omega}_n$ is contained in Ω_{n+1}. For each n we define the form $\mathscr{E}_n$ by

$$\mathscr{D}[\mathscr{E}_n] = \mathscr{F}_0^2(\Omega_n; \nu)$$

$$\mathscr{E}_n(u,v) = \sum_{i,j=1}^{d} \int_{\Omega_n} D_i u D_j v \, a_{ij} d\nu_{ij},$$

where $D_i = \mathbf{D}_i$ for $1 \leqq i \leqq d-1$. Then Step 1 says that for each n there is a diffusion X_n on Ω_n, which is uniquely associated with $\mathscr{E}_n$. Considering the method of construction of X_n, and using Fukushima's result again, we see that a part of X_{n+1} on Ω_n is equivalent to X_n. Therefore we invoke [2] to get a unique diffusion process X on Ω such that part of X on Ω_n is equivalent to X_n for all n. Of course X must be associated with $\mathscr{E}$. Since the property (i) of the resolvent R_λ follows from the proof of Lemma 7, the proof of the theorem is completed. q.e.d.

4. Polar Sets

In this section we are concerned with polar sets of our diffusion process.

Given two data $\{a_{ij}^{(1)}\}$, $\{a_{ij}^{(2)}\}$ satisfying the conditions (2) and (3), let $X^{(1)}$, $X^{(2)}$ be the corresponding diffusions of Theorem. Then we have

Theorem 10. *A set $N \subset \Omega$ is polar with respect to $X^{(1)}$ if and only if so it is with respect to $X^{(2)}$.*

Proof. In the case of Theorem 9 or Step 1 of the proof of the Theorem we have a resolvent density by Lemma 6, so that a set is polar if and only if its capacity is zero [5; Theorem 4]. But we see trivially that capacities corresponding to the given two data are equivalent. Hence we get the conclusion. In the general case there is a sequence $\{\Omega_n\}$ of bounded sets such that $X_n^{(i)}$, a part of $X^{(i)}$ on Ω_n, corresponds to Step 1 of the proof of Theorem for $i=1,2$. In order that N is polar with respect to $X^{(i)}$, it is necessary and sufficient that $N \cap \Omega_n$ is polar with respect to $X_n^{(i)}$ for every n satisfying $N \cap \Omega_n \neq \phi$. Therefore the desired assertion follows from the former case. q.e.d.

Since $\mathscr{F}_0^2(\Omega; \nu) \subset W_0^{1,2}(\Omega)$ (=Sobolev space) under condition (6) [6; Example 4.4.1] guarantees the following

Corollary 11. *Any polar set of our diffusion is also polar with respect to the Brownian motion.*

It is now our intention to show that the converse is not necessarily true. Namely that it is possible that our diffusion with a $(d-2)$-dimensional plane is not polar. To do this we need to study the structure of a Dirichlet space which is associated with a direct product motion of one-dimensional diffusions.

Let $X^{(i)}=[x_t^{(i)}, P_\cdot^{(i)}]$, $i=1, \ldots, d-1$ be independent one-dimensional Brownian motions on R^1 and $X^{(d)}=[x_t^{(d)}, P_\cdot^{(d)}]$ be a conservative one-dimensional diffusion on R^1 with the generator $G^{(d)}=\dfrac{d}{dn}\dfrac{d}{da}$. Here n is a bounded Borel measure on R^1 such that

$$
\begin{aligned}
&\int_{-\infty}^{0} da \int_{a}^{0} n(db) = +\infty, \quad \int_{-\infty}^{0} n(da) \int_{a}^{0} db < \infty, \\
&\int_{0}^{+\infty} da \int_{0}^{a} n(db) = +\infty, \quad \int_{0}^{+\infty} n(da) \int_{0}^{a} db < \infty.
\end{aligned} \tag{38}
$$

These conditions mean that the boundaries $\pm\infty$ are entrance, but not exit [10; §4.6]. $X^{(d)}$ has transition probability densities $p(t,a,b)$ such that

$$
\begin{cases}
0 \leqq p(t,a,b) = p(t,b,a) \\
\text{is continuous on } (0,+\infty) \times (-\infty,+\infty) \times (-\infty,+\infty), \\
\dfrac{\partial}{\partial t} p(t,a,b) = G^{(d)} p(t,a,b), \\
p_2^+(t,-\infty,b) = p_2^-(t,+\infty,b) = p_3^+(t,a,-\infty) = p_3^-(t,a,+\infty) = 0,
\end{cases} \tag{39}
$$

where $G^{(d)}$ is applied to a or to b and $p_2^+(t,a,b)=\lim_{h\downarrow 0}\{p(t,a+h,b)-p(t,a,b)\}/h$, $p_3^-(t,a,b)=\lim_{h\downarrow 0}\{p(t,a,b)-p(t,a,b-h)\}/h$ etc. (see [10; §4.11]).

Put

$$X^0=\left[x_t^0=(x_t^{(1)},\dots,x_t^{(d)}),\ P_x^0=\prod_{i=1}^{d}P_{x_i}^{(i)}\right],\qquad x=(x_1,\dots,x_d)\in R^d.$$

We denote by $(\mathscr{F}^0,\mathscr{E}^0)$ the Dirichlet space associated with X^0. As can easily be seen, in the same way as [8; Example 4.1], $(\mathscr{F}^0,\mathscr{E}^0)$ is related to $L^2(R^d;dv^0)$ and

$$C_0^\infty(R^d)\subset\mathscr{F}^0, \tag{40}$$

$$\mathscr{E}^0(u,v)=\frac{1}{2}\left(\sum_{i=1}^{d-1}\int_{R^d}D_iuD_iv\,dv^0+\int_{R^d}D_duD_dv\,dx\right)\qquad u,v\in C_0^\infty(R^d), \tag{41}$$

where $v^0(dx_1\dots dx_d)=dx_1\dots dx_{d-1}\,n(dx_d)$. Moreover we find that:

Lemma 12. *$(\mathscr{F}^0,\mathscr{E}^0)$ is regular, that is, $C_0^\infty(R^d)$ is dense in $\mathscr{F}^0$.*

Proof. Let R_λ^0 [resp. $R_\lambda^{(d)}$] denote the resolvent of X^0 [resp. $X^{(d)}$] and introduce a new function space:

$$\mathscr{H}=\left\{\sum_{i=1}^{l}\varphi_i(x')\,R_1^{(d)}\psi_i(x_d):\ \varphi_i\in C_0^\infty(R^{d-1}),\psi_i\in C_0^\infty(R^1),\ l\geqq 1\right\}.$$

Since

$$|||u|||^2_{2,R^d,\nu^0}/2(d+1)\leqq\mathscr{E}_1^0(u,u)\leqq|||u|||^2_{2,R^d,\nu^0},\qquad u\in C_0^\infty(R^d)$$

and $R_1^0(\mathscr{H})$ is a dense subspace of $\mathscr{F}^0$, it is sufficient to verify

$$R_1^0(\mathscr{H})\subset\mathscr{F}_0^2(R^d;\nu^0). \tag{42}$$

In fact (40) and (41) assert

$$\mathscr{F}_0^2(R^d;\nu^0)\subset\mathscr{F}^0,$$

$$\mathscr{E}_1^0(u,u)=\frac{1}{2}\left(\sum_{i=1}^{d-1}\int_{R^d}(\mathbf{D}_iu)^2\,dv^0+\int_{R^d}(D_du)^2\,dx\right)+\int_{R^d}u^2\,dv^0\qquad u\in\mathscr{F}_0^2(R^d;\nu^0);$$

on the other hand it follows from (42) that $\mathscr{F}^0\subset\mathscr{F}_0^2(\mathrm{R}^d;\nu^0)$, concluding $\mathscr{F}^0=\mathscr{F}_0^2(R^d;\nu^0)$.

Fix any $u(x',x_d)\equiv\varphi(x')\,R_1^{(d)}\psi(x_d)\in\mathscr{H}$ and set $v=R_1^{(d)}\psi$. Then

$$R_1^0u(x)\equiv R_1^0u(x',x_d)=\int_{R^d}g(x_d,y',y_d)\,\varphi(x'+y')\,v(y_d)\,dy'\,n(dy_d),$$

$$g(x_d,y',y_d)=\int_0^\infty e^{-t}(2\pi t)^{-(d-1)/2}\,e^{-|y'|^2/2t}\,p(t,x_d,y_d)\,dt.$$

Therefore for $1\leqq i\leqq d-1$

$$\begin{aligned}(\mathbf{D}_iR_1^0u(x))^2&=\left(\int_{R^d}g(x_d,y',y_d)\,D_i\varphi(x'+y')\,v(y_d)\,dy'\,n(dy_d)\right)^2\\&\leqq\int_{R^d}g(x_d,y',y_d)\,(D_i\varphi(x'+y')\,v(y_d))^2\,dy'\,n(dy_d),\end{aligned}$$

and so

$$\begin{aligned}&\|\mathbf{D}_i(R_1^0 u);\, L^2(R^d; dv^0)\|^2\\ &\leqq \int_{R^{d-1}} (D_i\varphi(x'))^2\, dx' \int_{R^1} v(y_d)^2\, n(dy_d) \int_{R^d} g(x_d, y', y_d)\, n(dx_d)\, dy'\\ &\leqq \|D_i\varphi;\, L^2(R^{d-1}; dx')\|^2\, \|v;\, L^2(R^1; dn)\|^2 < \infty.\end{aligned}$$

By (39) we obtain the following in distribution sense:

$$\begin{aligned}D_d R_1^0 u(x) = \int_{R_d} \varphi(x'+y')\, v(y_d)\, dy'\, n(dy_d) \int_0^\infty e^{-t}(2\pi t)^{-(d-1)/2}\\ \cdot e^{-|y'|^2/2t} D_{x_d} p(t, x_d, y_d)\, dt.\end{aligned}$$

By using (39) once more, we have also

$$\begin{aligned}\int_{-\infty}^{+\infty} D_a p(t, a, b)\, v(b)\, n(db) &= \int_{(-\infty, a]} n(db) \int_{-\infty}^{+\infty} p(t, b, c)\, G^{(d)} v(c)\, n(dc), \quad \text{or}\\ &= -\int_{[a, +\infty)} n(db) \int_{-\infty}^{+\infty} p(t, b, c)\, G^{(d)} v(c)\, n(dc).\end{aligned}$$

Since $|G^{(d)} v| = |R_1^{(d)}\psi - \psi| \leqq 2\sup|\psi|$,

$$\begin{aligned}(D_d(R_1^0 u(x)))^2 &\leqq \int_{R^{d-1}} \varphi(x'+y')^2\, dy' \int_0^\infty e^{-t}(2\pi t)^{-(d-1)/2} e^{-|y'|/2t}\, dt\\ &\times 4\sup|\psi|^2 \{n((-\infty, x_d])\, I_{(-\infty, 0]}(x_d) + n([x_d, +\infty))\, I_{[0, +\infty)}(x_d)\}^2.\end{aligned}$$

Hence

$$\begin{aligned}&\|D_d(R_1^0 u);\, L^2(R^d; dx)\|^2\\ &\leqq 4\sup|\psi|^2\, \|\varphi;\, L^2(R^{d-1}; dx')\|^2 n((-\infty, +\infty)) \left\{ \int_{-\infty}^0 n((-\infty, x_d])\, dx_d \right.\\ &\left. + \int_0^{+\infty} n([x_d, +\infty))\, dx_d \right\}\\ &< \infty.\end{aligned}$$

By the same method

$$\sum_{i=1}^{d-1} \|D_d \mathbf{D}_i(R_1^0 u);\, L^2(R^d; dx)\|^2 < \infty.$$

Thus $R_1^0 u$ satisfies the conditions of Proposition 1-(i), which leads us to $R_1^0 u \in \mathscr{F}_0^2(R^d; v^0)$. q.e.d.

Let k be an integer satisfying $0 \leqq k \leqq d$. $\kappa = (\kappa_1, \ldots, \kappa_k)$ denotes a k-tuple of integers satisfying $1 \leqq \kappa_1 < \ldots < \kappa_k \leqq d$. Given $a = (a_1, \ldots, a_d) \in \Omega$, let $E_\kappa(a)$ denote the k-dimensional plane in R^d, which contains a and is parallel to the plane spanned by coordinate axes corresponding to $x_{\kappa_1}, \ldots, x_{\kappa_k}$: $E_\kappa(a) = \{x \in R^d : x_i = a_i$ if

$i \notin \kappa\}$. In particular $E_\kappa(a)=\{a\}$ if $k=0$. Let $X=[x_t, \zeta, P_x]$ $(x \in \Omega)$ be a diffusion process of the Theorem.

Theorem 13. *For $E_\kappa(a) \cap \Omega$ to be polar it is necessary and sufficient that (i) $k \leqq d-3$, or (ii) $k=d-2$ and $\kappa \ni d$, or (iii) $k=d-2$, $\kappa \not\ni d$ and* $\int_{0+} \left\{ \int_0^t m((a_d-s, a_d+s))\, ds \right\}^{-1/2} dt = \infty$.

Proof. It is enough to provide a proof in the case of Theorem 9. Given a point a, put $H=H(a,1)$. Introduce a new positive Radon measure n on R^1, which coincides with m on (a_d-1, a_d+1) and satisfies (38). Let us consider the above mentioned direct product diffusion $X^0=[x_t^0, P_x^0]$ $(x \in R^d)$ corresponding to this n. Let X_H and X_H^0 denote parts of X and X^0 on H, respectively. Noting Lemma 12 and [6; Theorem 4.4.2] we find that both Dirichlet spaces associated with X_H and X_H^0 are regular. Furthermore we notice that X and X^0 have resolvent densities. Combining (10), [5; Theorem 4] and [6; Theorem 4.4.2] with these facts, we find that $E_\kappa(a)$ is polar if and only if the capacity of $E_\kappa(a) \cap H$ with respect to X_H (and hence X_H^0) is zero; moreover equivalently the capacity of $E_\kappa(a)$ with respect to X^0 is zero. Thus these statements are equivalent to $P_x^0(\sigma^0_{E_\kappa(a)} < \infty)=0$, $x \in R^d$, where $\sigma^0_{E_\kappa(a)}=\inf\{t>0: x_t^0 \in E_\kappa(a)\}$. It is evident that $P_x^0(\sigma^0_{E_\kappa(a)} < \infty)=1$, $x \in R^d$ if $k \geqq d-1$, and $P_x^0(\sigma^0_{E_\kappa(a)} < \infty)=0$, $X \in R^d$ if $k \leqq d-3$ or if $k=d-2$ and $\kappa \ni d$. Therefore we may assume that $k=d-2$ and $\kappa \not\ni d$, which reduces to the case that $d=2$ and $k=0$. However the necessary and sufficient condition in order to $P_x^0(\sigma^0_{\{a\}} < \infty)=0$, $x \in R^2$ is that the integral $\int_{0+} \left\{ \int_0^t m((a_2-s, a_2+s))\, ds \right\}^{-1/2} dt$ diverges, as we have already seen in [15; Theorem 5.1]. q.e.d.

Example 14. Let $0 \in \Omega \subset R^d$. Assume that $m((-t,t))=const. \times t^\beta$ as $t \downarrow 0$ with $0<\beta<1$. Then for $1 \leqq i \leqq d-1$ and $\alpha \in R^1$ the plane $S=\{x \in \Omega: x_i=\alpha,\ x_d=0\}$ cannot be polar with respect to our diffusion as long as it is not empty. But it is polar with respect to the Brownian motion.

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Loi de l'indice du lacet Brownien, et distribution de Hartman-Watson

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1. Introduction

Dans tout cet article, τ désigne un nombre positif fixé, z_0 est un nombre complexe, non nul, et $a=|z_0|$.

Soit $\mathbb{P}_{z_0}$ la loi du mouvement brownien complexe $(Z_t=X_t+iY_t)_{t\geq 0}$ issu de z_0, définie sur l'espace $\Omega_{\mathbb{C}}=C(\mathbb{R}_+,\mathbb{C})$, muni de sa tribu borélienne $\mathcal{F}$ (pour la topologie de la convergence uniforme sur tout compact de $\mathbb{R}_+$). On note encore, pour tout $z=x+iy\in\mathbb{C}$, $\mathbb{P}^z_{z_0}$ la loi du «pont brownien complexe», entre 0 et τ, issu de z_0 en $t=0$, et aboutissant en z, au temps $t=\tau$. Rappelons que:

$$\mathbb{P}_{z_0}|\mathcal{F}_\tau=\iint \frac{dx\,dy}{(2\pi\tau)}\,e^{-\frac{(x-x_0)^2+(y-y_0)^2}{2\tau}}\,\mathbb{P}^z_{z_0} \tag{1.1}$$

où $\mathcal{F}_\tau=\sigma\{Z_s, 0\leq s\leq\tau\}$, égalité que l'on peut réécrire en:

$$\text{pour tout } A\in\mathcal{F}_\tau,\quad \mathbb{P}_{z_0}(A\,|\,Z_\tau=z)=\mathbb{P}^z_{z_0}(A) \tag{1.1'}$$

(tous les rappels nécessaires, concernant le pont brownien- réel ou complexe- sont faits au paragraphe 2).

Paul Lévy a montré que le point 0 n'est pas visité par le processus $(Z_t)_{t\geq 0}$, $\mathbb{P}_{z_0}$ p.s. On déduit donc de (1.1), ou (1.1'), que:

$$dx\,dy \text{ p.s.},\quad \mathbb{P}^z_{z_0}(\exists\, t\in[0,\tau], Z_t=0)=0, \tag{1.2}$$

et, en fait, on montre au paragraphe 2, plus précisément, que:

$$\text{pour tout } z\neq 0,\quad \mathbb{P}^z_{z_0}(\exists\, t\in[0,\tau], Z_t=0)=0. \tag{1.2'}$$

On peut donc définir, par continuité, pour tout $t\in[0,\tau]$, l'accroissement de l'argument entre 0 et t, soit: $\theta_t(\omega)-\theta_0(\omega)$, de la courbe $(Z_u(\omega), u\in[0,t])$, et ceci peut être fait, d'après ce qui précède $\mathbb{P}^z_{z_0}$ p.s., pour tout $z\neq 0$ (et donc $\mathbb{P}_{z_0}$ p.s.!).

L'un des buts de cet article est de calculer la loi de $(\theta_\tau-\theta_0)$, pour $\mathbb{P}^z_{z_0}$, pour tout $z\neq 0$, et en particulier pour $z=z_0$, auquel cas, sous $\mathbb{P}^*_{z_0}\overset{\text{déf}}{=}\mathbb{P}^{z_0}_{z_0}$, le processus

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$(Z_t)_{0\leq t\leq\tau}$ décrit une boucle, que l'on appelle lacet brownien, et l'on a: $(\theta_\tau-\theta_0)\in 2\pi\mathbb{Z}$, $\mathbb{P}^*_{z_0}$ p.s.

Par ailleurs, on rappelle, au paragraphe 3, que $(\theta_t-\theta_0)$ est une $\mathbb{P}_{z_0}$ martingale locale continue, dont le processus croissant est égal à $\int_0^t \frac{ds}{|Z_s|^2}$, et les calculs menés ci-dessous permettent en fait d'obtenir, de façon explicite, la loi conjointe de $\left(\theta_\tau-\theta_0; \int_0^\tau \frac{ds}{|Z_s|^2}\right)$ sous $\mathbb{P}^*_{z_0}(z\neq 0)$ (voir le paragraphe 6).

Une étape importante du travail est constituée par l'obtention de la loi, sous $\mathbb{P}_{z_0}$, de $(\theta_\tau-\theta_0)$, conditionnellement à $|Z_\tau|$, loi qui est, précisément, donnée par la formule suivante, valable pour tout $\nu\in\mathbb{R}$:

$$\mathbb{E}_{z_0}(e^{i\nu(\theta_\tau-\theta_0)}\,|\,|Z_\tau|=\rho)=\frac{I_{|\nu|}\left(\frac{\rho a}{\tau}\right)}{I_0\left(\frac{\rho a}{\tau}\right)}, \tag{1.3}$$

où I_ν désigne la fonction de Bessel modifiée, du premier ordre, et d'indice ν.

Ce résultat donne une interprétation probabiliste des distributions μ_r et η_r, définies pour tout $r>0$, respectivement par leur transformée de Fourier (resp. Laplace):

$$\nu\to\frac{I_{|\nu|}(r)}{I_0(r)} \quad\text{et}\quad \lambda\to\frac{I_{\sqrt{2\lambda}}(r)}{I_0(r)}.$$

μ_r(resp: η_r) est désignée, dans la littérature, comme la distribution développée de von Mises - en anglais: «unwrapped von Mises distribution» -, resp.: distribution de Hartman-Watson (cf. [4]; 1974).

Les mesures μ_r et η_r sont absolument continues par rapport à la mesure de Lebesgue (respectivement sur $\mathbb{R}$ et $\mathbb{R}_+$); leurs densités sont étudiées au paragraphe 5.

La méthode qui nous a conduit à la formule (1.3) permet également - c'est là le second aspect de ce travail - de redémontrer (cf., paragraphe 4) des résultats plus récents de Hartman ([3]; 1976), à savoir:

(1.4) pour tout couple (r, R) vérifiant: $0<r<R<\infty$, les fonctions suivantes, en $\lambda\in\mathbb{R}_+$:

$$\frac{I_{\sqrt{2\lambda}}(r)}{I_0(r)}\times\frac{I_0(R)}{I_{\sqrt{2\lambda}}(R)} \quad\text{et}\quad \frac{K_{\sqrt{2\lambda}}(R)}{K_0(R)}\times\frac{K_0(r)}{K_{\sqrt{2\lambda}}(r)}$$

sont les transformées de Laplace de probabilités sur $\mathbb{R}_+$.

Le paragraphe 6 est consacré à expliciter la loi de l'indice du lacet brownien défini plus haut, tandis qu'au paragraphe 7, on développe, en dimension supérieure à 2, certains des résultats obtenus précédemment en dimension 2.

2. Quelques précisions sur les ponts browniens réel et complexe

Dans le seul but de simplifier les expressions qui interviennent dans ce paragraphe, on suppose le temps τ égal à 1; les arguments utilisés sont également valables lorsque τ est un réel strictement positif.

Soit, sur un espace de probabilité $(\Omega, \mathfrak{A}, P)$, un mouvement brownien réel $(B_t)_{t\geq 0}$, issu de x_0. On note $(\mathfrak{B}_t)_{t\geq 0}$ sa filtration naturelle, et

$$\tilde{\mathfrak{B}}_t = \bigcap_{(\varepsilon>0)} (\mathfrak{B}_{(t+\varepsilon)} \vee \sigma\{B_1\})$$

K. Ito ([5]; voir aussi [7]) a remarqué que le processus $(B_t)_{t\leq 1}$ est une $(\tilde{\mathfrak{B}}_t)$ quasi-martingale, dont la décomposition canonique s'écrit:

$$B_t = \tilde{B}_t + \int_0^t \frac{(B_1 - B_s)}{1-s}\, ds \quad (t \leqq 1). \tag{2.1}$$

Soulignons que, d'une part:

(2.2) $(\tilde{B}_t)_{t\leq 1}$ *est un* $(\tilde{\mathfrak{B}}_t)$ *mouvement brownien, issu de* x_0, et, en conséquence:

(2.3) $(\tilde{B}_t)_{t\leq 1}$ *est indépendant de* (B_1);

d'autre part:

$$E \int_0^1 \frac{|B_1 - B_s|}{(1-s)}\, ds < \infty. \tag{2.4}$$

Par ailleurs, si $x \in \mathbb{R}$, on considère l'équation en U:

$$e(x_0, x): \qquad U_t = \tilde{B}_t + \int_0^t \frac{x - U_s}{(1-s)}\, ds \quad (t<1),$$

dont l'unique solution s'explicite aisément:

$$U_t = xt + \tilde{B}_t - (1-t) \int_0^t \frac{\tilde{B}_u\, du}{(1-u)^2} \quad (t<1). \tag{2.5}$$

De plus, on a le:

(2.6) **Lemme.** *La solution* (U_t) *de* $e(x_0, x)$ *vérifie:* $\lim_{t\uparrow\uparrow 1} U_t = x$, P p.s.

Démonstration. D'après (2.5), il s'agit de montrer que:

$$\tilde{B}_t - (1-t) \int_0^t \frac{\tilde{B}_u\, du}{(1-u)^2} \xrightarrow[t\uparrow\uparrow 1]{} 0,$$

ou encore que:

$$H_t \equiv \tilde{B}_1\, t - (1-t) \int_0^t \frac{\tilde{B}_u\, du}{(1-u)^2} \xrightarrow[t\uparrow\uparrow 1]{} 0.$$

Or,

$$H_t = (1-t) \int_0^t \frac{(\tilde{B}_1 - \tilde{B}_u)\, du}{(1-u)^2}.$$

Rappelons le résultat élémentaire suivant ([7], Lemme 9): si $g: [0,1] \to \mathbb{R}_+$ est une fonction continue décroissante, telle que $g(1)=0$, alors, pour toute fonction $f \geqq 0$ vérifiant $\int_0^1 f(u)\, g(u)\, du < \infty$, on a

$$\lim_{t\uparrow\uparrow 1} g(t) \int_0^t f(u)\, du = 0.$$

D'après (2.4), ceci s'applique, $P(d\omega)$ p.s., à $g(t)=(1-t)$, et

$$f(t)=\frac{|\tilde{B}_1-\tilde{B}_t|(\omega)}{(1-t)^2}. \quad \square$$

Sur l'espace $\Omega_{\mathbb{R}}=C(\mathbb{R}_+,\mathbb{R})$, on considère le processus des projections $X_t(\omega)=\omega(t)$ $(t\geqq 0)$, la famille de tribus $\mathfrak{G}_t=\sigma\{X_s, s\leqq t\}$ $(t\geqq 0)$, et $\mathfrak{G}=V_t\mathfrak{G}_t$.

P_{x_0} désigne la loi, sur $(\Omega_{\mathbb{R}}, \mathfrak{G})$, du mouvement brownien réel, issu de x_0. Pour tout $x\in\mathbb{R}$, notons $P^x_{x_0}$ la distribution, sur $(\Omega_{\mathbb{R}}, \mathfrak{G}_1)$, de la solution $(U_t)_{t<1}$ de $e(x_0, x)$, prolongée par continuité en $t=1$ (donc $U_1=x$, d'après le lemme (2.6)).

D'après (2.1) et (2.3), on a:

(2.7) *l'application: $x\to P^x_{x_0}$ est une désintégration régulière de P_{x_0}, par rapport à la tribu engendrée par X_1.*

Pour tout couple (x_0, x), on note $PB(x_0, x)$ le triplet $(\Omega_{\mathbb{R}_+}, (X_t)_{t\leqq 1}, P^x_{x_0})$: on a ainsi défini, d'après (2.6) et (2.7), le pont brownien issu de x_0, et aboutissant en x au temps $t=1$.

Remarque. La présentation de $PB(x_0, x)$ adoptée ici est semblable à celle de Mackevicius ([12]), mais les démonstrations sont différentes. $\square$

Pour tout couple $((x_0, x); (y_0, y))$ de points de $\mathbb{R}^2$, on note $z_0=x_0+iy_0$, $z=x+iy$, et on définit, sur $(\Omega_{\mathbb{C}}, \mathfrak{Z}_1)$, $\mathbb{P}^z_{z_0}=P^x_{x_0}\otimes P^y_{y_0}$ (modulo les identifications canoniques!): c'est, par définition, la distribution du pont brownien complexe $PB(z_0, z)$ issu de z_0, et aboutissant en z, au temps $t=1$; d'après (2.6) et (2.7), l'égalité (1.1) est réalisée, ainsi que: $\mathbb{P}^z_{z_0}(Z_1=z)=1$, pour tout z.

De plus, on a le:

(2.8) **Lemme.** *Pour tout couple $(z_0, z)\in\mathbb{C}^2$, $\mathbb{P}^z_{z_0}(\exists\, t\in]0,1[, Z_t=0)=0$.*

Démonstration. 1) (X_t) et (Y_t) désignent respectivement les parties réelle et imaginaire de (Z_t).

Définissons les $(\mathfrak{Z}_t)$ temps d'arrêt:

$$T_n(\omega)\equiv\inf\left\{t/A^z_t\equiv\int_0^t\frac{(x-X_s)^2+(y-Y_s)^2}{(1-s)^2}\,ds\geqq n\right\}.$$

Admettons provisoirement que:

$$\mathbb{P}^z_{z_0}\text{ p.s., } T_n\uparrow\uparrow 1. \tag{2.9}$$

Définissons, en calquant les notations sur celles du début de ce paragraphe, les processus:

$$\tilde{X}_t\equiv X_t-\int_0^t\frac{(x-X_s)}{(1-s)}\,ds \quad (t<1),$$

$$\tilde{X}_1\equiv\lim_{t\uparrow\uparrow 1}\tilde{X}_t \text{ (qui existe } \mathbb{P}^z_{z_0}\text{ p.s., par définition de } PB(x_0, x)),$$

et $(\tilde{Y}_t)_{t\leqq 1}$ de façon analogue.

Par définition de $\mathbb{P}^z_{z_0}$, $\tilde{Z}\equiv((\tilde{X}_t+i\tilde{Y}_t)_{t\leqq 1})$ est un $(\mathbb{P}^z_{z_0}, (\mathfrak{Z}_t)_{t\leqq 1})$ mouvement brownien complexe issu de z_0.

D'après le théorème de Girsanov, pour tout $n \geqq 1$, le processus $(Z_{t \wedge T_n})_{t \geqq 0}$ est, sous la probabilité:

$$Q^{(n)} \equiv \exp\left\{ -\int_0^{T_n} \frac{[(x-X_s)\,d\tilde{X}_s + (y-Y_s)\,d\tilde{Y}_s]}{(1-s)} - \tfrac{1}{2} A^z_{T_n} \right\} \cdot \mathbb{P}^z_{z_0}$$

un mouvement brownien complexe, issu de z_0 en $t=0$, et arrêté en T_n.

Les probabilités $Q^{(n)}$ et $\mathbb{P}^z_{z_0}$ étant équivalentes, on a, d'après le théorème de P. Lévy déjà mentionné: pour tout $z \in \mathbb{C}$, $\mathbb{P}^z_{z_0}(\exists t \in]0, T_n],\ Z_t = 0) = 0$, d'où le résultat cherché, d'après (2.9).

2) Il reste à montrer (2.9), qui sera une conséquence immédiate de l'assertion:

(2.10) pour tout $(x_0, x) \in \mathbb{R}^2$, $\int_0^1 \frac{(x-X_s)^2}{(1-s)^2}\,ds = \infty$, $P^x_{x_0}$ p.s.

Or, d'après (2.5), on a, pour tout $s<1$:

$$\frac{x-X_s}{(1-s)} = (x-\tilde{X}_s) + \int_0^s \frac{(\tilde{X}_u - \tilde{X}_s)}{(1-u)^2}\,du.$$

Ainsi, puisque $(\tilde{X}_u)_{u \leqq 1}$ est, sous $P^x_{x_0}$, un mouvement brownien réel issu de x_0, (2.10) équivaut à:

(2.11) $$\int_0^1 \left\{ \int_0^s \frac{(X_u - X_s)\,du}{(1-u)^2} \right\}^2 ds = \infty \qquad P_{x_0} \text{ p.s.}$$

Montrons maintenant (2.10). D'après [7] (corollaire 3.2), on a:

$$\int_0^1 \frac{(X_1 - X_s)^2}{(1-s)^2}\,ds = \infty, \quad P_{x_0} \text{ p.s., et donc:}$$

$$dx \text{ p.s.}, \int_0^1 \frac{(x-X_s)^2\,ds}{(1-s)^2} = \infty, \quad P^x_{x_0} \text{ p.s.}$$

(2.10) est donc réalisée au moins pour un $x \in \mathbb{R}$, et donc, d'après l'équivalence de (2.10) et (2.11) (qui ne dépend pas de x!), pour tout $x \in \mathbb{R}$. □

Ainsi, si $z_0 z \neq 0$, on peut donc définir, comme il a été dit dans l'introduction, par continuité, pour $t \in [0,1]$, l'accroissement de l'argument entre 0 et t, soit: $\theta_t(\omega) - \theta_0(\omega)$, de la courbe $(Z_u(\omega), u \in [0,t])$, $\mathbb{P}^z_{z_0}$ p.s.

Le lemme suivant nous sera utile au paragraphe 6.

(2.12) **Lemme.** 1) *Soit $z_0 \in \mathbb{C}$, et $F: \Omega_{\mathbb{C}} \to \mathbb{R}$ une fonction bornée, continue pour la topologie de la convergence compacte, et $\mathcal{F}_1$-mesurable. Alors, l'application: $z \to \mathbb{E}^z_{z_0}(F)$ est continue.*

2) *Soit $z_0 \neq 0$, et $f: \mathbb{R} \to \mathbb{R}$ une fonction continue, bornée. Alors, l'application: $z \to \mathbb{E}^z_{z_0}[f(\theta_1 - \theta_0)]$ est définie, et continue, sur $\mathbb{C} \setminus \{0\}$.*

Démonstration. 1) Rappelons, avec les notations du lemme (2.8), que:

$$Z_t = zt + \tilde{Z}_t - (1-t)\int_0^t \frac{\tilde{Z}_u\,du}{(1-u)^2}, \quad (t<1)$$

et que $(\hat{Z}_t)_{t\leq 1}$ est, sous $\mathbb{P}^z_{z_0}$, un mouvement brownien complexe issu de z_0. Ainsi, si l'on définit la famille d'applications $(\gamma_z(\omega))$, de $\mathbb{R}_+$ dans $\mathbb{C}$, par:

$$[\gamma_z(\omega)](t)=\begin{cases} zt+Z_t(\omega)-(1-t)\int_0^t \dfrac{Z_u(\omega)\,du}{(1-u)^2} & (t<1) \\ z & (t\geqq 1), \end{cases}$$

alors, on a: $\mathbb{E}^z_{z_0}[F(\omega)]=\mathbb{E}_{z_0}[F(\gamma_z(\omega))]$, et la première partie du lemme découle de ce que, pour tout $\omega\in\Omega_{\mathbb{C}}$, l'application: $\mathbb{C}\to\Omega_{\mathbb{C}}$ est continue, $\Omega_{\mathbb{C}}$ étant

$$z\to\gamma_z(\omega)$$

muni de la topologie de la convergence compacte.

2) Supposons $z_0\neq 0$. Alors, pour tout $z\neq 0$, l'application

$$\text{Arg}:\omega\to\theta_1(\omega)-\theta_0(\omega)\equiv \text{Im}\left\{\int_{\omega([0,1])}\frac{du}{u}\right\}$$

est définie sur l'ouvert $\mathbf{0}=\{\omega/\forall t\in[0,1],\omega(t)\neq 0\}$ de $\Omega_{\mathbb{C}}$, qui est, d'après le lemme (2.8), de mesure pleine pour $\mathbb{P}^z_{z_0}$; de plus, Arg est continue sur $\mathbf{0}$. La seconde partie du lemme découle donc de la première.

3. Parties radiale et angulaire du mouvement brownien complexe, sous $\mathbb{P}_{z_0}$

Dorénavant, z_0 est un nombre complexe non nul. Ce paragraphe est une suite de rappels succincts, extraits de [16] (voir aussi Getoor-Sharpe [1]). Une application simple de la formule d'Ito montre que, sous $\mathbb{P}_{z_0}$:

$$Z_t=Z_0\exp\left(\int_0^t\frac{dZ_s}{Z_s}\right)^1, \tag{3.1}$$

ce qui équivaut à:

$$\log_\omega(Z_t(\omega))-\log_\omega(Z_0(\omega))=\int_0^t\frac{dZ_s}{Z_s}, \tag{3.2}$$

où $\log_\omega$ désigne une détermination continue du logarithme le long de la trajectoire $Z_\cdot(\omega)$.

Identifions respectivement les parties réelle et imaginaire des deux membres de (3.2). Il vient, en notant $\rho_t=|Z_t|$:

$$\log\rho_t-\log\rho_0=\int_0^t\frac{d\beta_s}{\rho_s}; \tag{3.3}$$

$$\theta_t-\theta_0=\int_0^t\frac{d\gamma_s}{\rho_s}, \tag{3.4}$$

---

[1] Contrairement à ce qui peut sembler au premier abord, cette formule n'est pas vraie pour toute martingale complexe, continue (Z_t), qui ne visite pas $\{0\}$; en fait, une telle martingale complexe (Z_t) vérifie (3.1) si, et seulement si, elle est conforme (c'est à dire: Z^2 est également une martingale locale)

où

$$\beta_t = \int_0^t \frac{(X_s dX_s + Y_s dY_s)}{\rho_s}, \quad \text{et} \quad \gamma_t = \int_0^t \frac{(X_s dY_s - Y_s dX_s)}{\rho_s}$$

sont, d'après la caractérisation de Paul Lévy, deux mouvements browniens réels indépendants, puisque: $\langle \beta, \beta \rangle_t = \langle \gamma, \gamma \rangle_t = t$, et $\langle \beta, \gamma \rangle_t = 0$.

D'autre part, une nouvelle application de la formule d'Ito à $\rho_t \equiv (X_t^2 + Y_t^2)^{1/2}$ montre que:

$$\rho_t = \rho_0 + \beta_t + \tfrac{1}{2} \int_0^t \frac{ds}{\rho_s}. \tag{3.5}$$

La fonction: $x \to 1/x$ étant différentiable sur $]0, \infty[$, et (ρ_t) ne s'annulant pas, on déduit aisément de (3.5) que:

(3.6) *les processus* (ρ_t) *et* (β_t) *ont même filtration.*

Ainsi, d'après (3.4), et l'indépendance de (β_t) et (γ_t), on a:

(3.7) *sous* $\mathbb{P}_{z_0}$, *conditionnellement à* $\rho = (\rho_t, t \geqq 0)$, *le processus* $(\theta_t - \theta_0)$ *est une martingale gaussienne, centrée, de processus croissant* $\int_0^t \frac{ds}{\rho_s^2}$.

Nota bene: Le résultat précédent peut aussi être obtenu en exprimant $\frac{1}{2}\Delta = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ en coordonnées polaires (cf., Ito-McKean [6], p. 270), mais la présentation ci-dessus est cohérente avec l'esprit de l'article, et sera encore utile par la suite.

4. Fonctions, et processus, de Bessel

Ce paragraphe est consacré à la démonstration des résultats (1.4) de Hartman, ainsi qu'à l'obtention de la formule (1.3).

Introduisons tout d'abord quelques notations relatives aux processus de Bessel: on appelle processus de Bessel d'indice $\nu (\in \mathbb{R})$, et on note BES(ν), la diffusion à valeurs dans $[0, \infty)$, de générateur infinitésimal:

$$A_\nu = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu + 1}{2x}\frac{d}{dx} \tag{4.1}$$

(pour une définition plus complète, voir J. Kent [10]).

On désigne quelquefois $d_\nu \equiv 2(\nu + 1)$ comme la «dimension» de BES(ν): en effet, la partie radiale du mouvement brownien à valeurs dans $\mathbb{R}^d$ est le processus BES $\left(\frac{d}{2} - 1\right)$. Dans la suite, l'indice $\nu = 0$, c'est à dire $d_\nu = 2$, joue un rôle fondamental.

Sur l'espace $\Omega_{\mathbb{R}_+}=C(\mathbb{R}_+,\mathbb{R}_+)$, on note $\rho_t(\omega)=\omega(t)$, $\mathfrak{F}_t=\sigma\{\rho_s, s\leqq t\}$ $(t\geqq 0)$, et $\mathfrak{F}=\sigma\{\rho_s, s\geqq 0\}$. Pour tout $a\in[0,\infty)$, P_a^ν désigne la distribution, sur $(\Omega_{\mathbb{R}_+},\mathfrak{F})$, de BES($\nu$), qui vérifie $P_a^\nu[\rho_0=a]=1$.

Remarquons que, d'après (4.1), pour tout $a>0$, et tout $\nu\geqq 0$:

$$\beta_t\equiv\rho_t-\rho_0-\frac{2\nu+1}{2}\int_0^t\frac{ds}{\rho_s} \tag{4.2}$$

est un $(P_a^\nu,\mathfrak{F}_t)$ mouvement brownien réel; ce résultat généralise la formule (3.5), qui correspond au cas $\nu=0$.

Les résultats suivants, dûs à J. Kent [10], jouent un rôle important dans le reste du paragraphe:

– pour tout $\nu>-1$, $x>0$, $t>0$, la fonction de transition $P_t^\nu(x,dy)$ de BES(ν) admet une densité $p_t^\nu(x,y)$ qui vérifie:

$$\left(\frac{y}{x}\right)^{-\nu}p_t^\nu(x,y)=\frac{1}{t}e^{-\frac{x^2+y^2}{2t}}I_\nu\left(\frac{xy}{t}\right)y \qquad (y>0) \tag{4.3}$$

(lorsque $\nu=\frac{d}{2}-1$, $d\in\mathbb{N}^*$, cette égalité figure dans Ito-Mc Kean [6], paragraphe (2.7)).

– pour tout $b>0$, notons $\tau_b=\inf\{t\geqq 0/\rho_t=b\}$. Alors, pour tout $\lambda>0$;

$$E_a^\nu[e^{-\lambda\tau_b}]=\left(\frac{b}{a}\right)^\nu\frac{\mathfrak{L}_\nu(a\sqrt{2\lambda})}{\mathfrak{L}_\nu(b\sqrt{2\lambda})}, \tag{4.4}$$

où $\mathfrak{L}=K$, si $\nu\in\mathbb{R}$, $0<b<a<\infty$, et $\mathfrak{L}=I$ si $\nu>-1$, $0<a<b<\infty$. (Voir aussi Getoor-Sharpe [2], proposition (5.7).)

La clé des théorèmes qui figurent dans ce paragraphe est le

(4.5) **Lemme.** *Soient a un réel strictement positif fixé, et T un $(\mathfrak{F}_{s+})$ temps d'arrêt borné. Alors, pour tout couple (μ,ν) d'indices positifs ou nuls, on a, pour toute variable positive Y, $\mathfrak{F}_{T+}$-mesurable:*

$$E_a^\mu\left[Y\exp\left\{-\frac{\nu^2}{2}\int_0^T\frac{ds}{\rho_s^2}\right\}\left(\frac{\rho_T}{a}\right)^{-\mu}\right]=E_a^\nu\left[Y\exp\left(-\frac{\mu^2}{2}\int_0^T\frac{ds}{\rho_s^2}\right)\left(\frac{\rho_T}{a}\right)^{-\nu}\right]. \tag{4.6}$$

Démonstration. 1) Démontrons tout d'abord (4.6) pour $\mu=0$, et $\nu>0$. (4.6) équivaut alors, à l'évidence, à l'identité:

$$E_a^0\left[Y\left(\frac{\rho_T}{a}\right)^\nu\exp\left(-\frac{\nu^2}{2}\int_0^T\frac{ds}{\rho_s^2}\right)\right]=E_a^\nu(Y), \tag{$*$}$$

pour toute variable positive Y, $(\mathfrak{F}_{T+})$ mesurable.

2) Considérons la $(P_a^0,\mathfrak{F}_t)$ martingale locale:

$$L_t^\nu\equiv\exp\left\{\nu\int_0^t\frac{d\beta_s}{\rho_s}-\frac{\nu^2}{2}\int_0^t\frac{ds}{\rho_s^2}\right\}$$
$$=\left(\frac{\rho_t}{a}\right)^\nu\exp\left\{-\frac{\nu^2}{2}\int_0^t\frac{ds}{\rho_s^2}\right\}, \quad \text{d'après (3.3).}$$

L'inégalité évidente: $L_t^\nu \leqq \left(\frac{\rho_t}{a}\right)^\nu$, et l'inégalité de Doob dans $L^p(1<p<\infty)$ entraînent que, pour tout $p\in]1,\infty[$, et $t>0$:

$$E\left[\sup_{s\leqq t}(L_s^\nu)^p\right]<\infty.$$

En particulier, pour tout $\nu>0$, $(L_t^\nu)_{t\geqq 0}$ est une $(P_a^0, \mathfrak{F}_t)$ *martingale.*

3) Soit σ une constante positive telle que: $T\leqq\sigma$. D'après 2), $Q_\sigma^\nu \equiv L_\sigma^\nu \cdot (P_a^0|\mathfrak{F}_\sigma)$ est une probabilité sur $(\Omega_{\mathbb{R}^+}, \mathfrak{F}_\sigma)$. D'après le théorème de Girsanov, $\tilde{\beta}_t \equiv \beta_t - \nu\int_0^t \frac{ds}{\rho_s}(t\leqq\sigma)$ est un $(Q_\sigma^\nu, (\mathfrak{F}_t)_{t\leqq\sigma})$ mouvement brownien réel.

On réécrit maintenant l'égalité (3.5) en:

$$\rho_t = a + \tilde{\beta}_t + \frac{2\nu+1}{2}\int_0^t \frac{ds}{\rho_s}, \tag{3.5}$$

ce qui, d'après (4.2), montre que, sous Q_σ^ν, le processus $(\rho_t)_{t\leqq\sigma}$ est un processus de Bessel d'indice ν.

On a donc, pour toute variable positive Y, $(\mathfrak{F}_{T^+})$ mesurable:

$$\begin{aligned} E_a^0\left[Y\left(\frac{\rho_T}{a}\right)^\nu \exp\left\{-\frac{\nu^2}{2}\int_0^T \frac{ds}{\rho_s^2}\right\}\right] &= E_a^0[Y L_T^\nu] \\ &\overset{(i)}{=} E_a^0[Y L_\sigma^\nu] \\ &= E_{Q_\sigma^\nu}(Y) = E_a^\nu(Y). \end{aligned}$$

(l'égalité (i) découle de ce que $(L_t^\nu)_{t\geqq 0}$ est une $(P_a^0, \mathfrak{F}_t)$ martingale). $(*)$ est donc démontrée.

4) Enfin, (4.6) est une conséquence de $(*)$, appliquée d'abord avec l'indice $\mu\geqq 0$, puis avec l'indice $\nu\geqq 0$. □

Remarque importante. Dans son article [8] (voir aussi, Kazamaki et Sekiguchi [9]), Kazamaki donne un exemple montrant que son critère:

$\boxed{\exp\{\tfrac{1}{2}M_t\} \text{ sous-martingale}}$ pour que $\exp\{M_t - \tfrac{1}{2}\langle M\rangle_t\}$ soit une *martingale* est meilleur que le critère, antérieur, de Novikov [13], à savoir:

$\boxed{\text{pour tout } t,\ E(\exp\tfrac{1}{2}\langle M\rangle_t)<\infty}$.

Remarquons que le critère de Novikov ne s'applique pas non plus à la martingale (locale) $L_t = \exp\{M_t - \frac{1}{2}\langle M\rangle_t\}$ qui figure dans la démonstration précédente, car:

$$\langle M\rangle_t = \nu^2\int_0^t \frac{ds}{\rho_s^2}, \quad \text{et} \quad E_a^0\left(\frac{1}{\rho_s^2}\right)\equiv\infty \quad (s>0).$$

Par contre, le critère de Kazamaki s'applique à l'évidence car: $\exp\left\{\frac{M_t}{2}\right\} = \left(\frac{\rho_t}{a}\right)^{\nu/2}$. Autrement dit, voici un autre exemple illustrant la puissance

du test de Kazamaki (qui, soulignons le, est loin d'avoir été utilisé dans toute sa force. □

Les deux théorèmes suivants constituent, à l'évidence, lorsque l'on prend $\mu=0$, une redémonstration probabiliste *directe*, respectivement du résultat de Hartman et Watson (voir l'introduction), et des résultats (1.4) de Hartman.

(4.7) **Théorème.** *Pour tout couple* $(\mu, \nu)\in\mathbb{R}_+\times\mathbb{R}$, *et* $\tau>0$, *on a:*

$$E_a^\mu\left[\exp\left\{-\frac{\nu^2}{2}\int_0^\tau\frac{ds}{\rho_s^2}\right\}\Big/\rho_\tau=\rho\right]=\frac{I_{(\mu^2+\nu^2)^{1/2}}\left(\frac{\rho a}{\tau}\right)}{I_\mu\left(\frac{\rho a}{\tau}\right)} \tag{4.8}$$

Démonstration. Appliquons (4.6), avec le couple d'indices (μ, λ), où $\lambda=(\mu^2+\nu^2)^{1/2}$, au temps $T=\tau$, et avec la variable:

$$Y=f(\rho_\tau)\exp\left\{\frac{\mu^2}{2}\int_0^\tau\frac{ds}{\rho_s^2}\right\},$$

f désignant ici une fonction borélienne positive. Il vient:

$$\begin{aligned}
&E_a^\mu\left[f(\rho_\tau)\exp\left(-\frac{\nu^2}{2}\int_0^\tau\frac{ds}{\rho_s^2}\right)\left(\frac{\rho_\tau}{a}\right)^{-\mu}\right]\\
&\quad=E_a^\lambda\left[f(\rho_\tau)\left(\frac{\rho_\tau}{a}\right)^{-\lambda}\right]\\
&\quad=\int_0^\infty dy f(y)\frac{1}{\tau}\exp\left(-\frac{a^2+y^2}{2\tau}\right)I_\lambda\left(\frac{ay}{\tau}\right)y \qquad \text{(d'après (4.3)).}
\end{aligned}$$

Désignons par G une version borélienne de $E_a^\mu\left[\exp\left\{-\frac{\nu^2}{2}\int_0^\tau\frac{ds}{\rho_s^2}\right\}\Big/\rho_\tau=\cdot\right]$. Il vient alors, en appliquant à nouveau (4.3):

$$\begin{aligned}
&\int_0^\infty dy f(y)G(y)\frac{1}{\tau}\exp\left(-\frac{a^2+y^2}{2\tau}\right)I_\mu\left(\frac{ay}{\tau}\right)y\\
&=\int_0^\infty dy f(y)\frac{1}{\tau}\exp\left(-\frac{a^2+y^2}{2\tau}\right)I_\lambda\left(\frac{ay}{\tau}\right)y, \qquad \text{d'où (4.8).}\ \square
\end{aligned}$$

Remarque. A l'aide de la propriété de Markov, on peut écrire la version symétrisée suivante de (4.8): pour tout couple $(\mu, \nu)\in\mathbb{R}_+\times\mathbb{R}$, et $0<\sigma<\tau<\infty$:

$$E_a^\mu\left[\exp\left(-\frac{\nu^2}{2}\int_\sigma^\tau\frac{ds}{\rho_s^2}\right)\Big/(\rho_\sigma,\rho_\tau)\right]=\frac{I_{(\mu^2+\nu^2)^{1/2}}\left(\frac{\rho_\sigma\rho_\tau}{\tau-\sigma}\right)}{I_\mu\left(\frac{\rho_\sigma\rho_\tau}{\tau-\sigma}\right)}$$

Le corollaire suivant découle immédiatement de (3.7).

(4.7.1) **Corollaire.** *Pour tout* $\alpha\in\mathbb{R}$, *on a:*

$$\mathbb{E}_{z_0}(e^{i\alpha(\theta_\tau-\theta_0)}/\rho_\tau=\rho)=E_a^0\left(\exp-\frac{\alpha^2}{2}\int_0^\tau\frac{ds}{\rho_s^2}\Big/\rho_\tau=\rho\right)=\frac{I_{|\alpha|}\left(\frac{\rho a}{\tau}\right)}{I_0\left(\frac{\rho a}{\tau}\right)}. \tag{4.9}$$

Remarque. Au début de ce travail, la formule «intégrée»:

$$\mathbb{E}_{z_0}(e^{i\alpha(\theta_\tau-\theta_0)})=\int\mathbb{P}_{z_0}(\rho_\tau\in d\rho)\frac{I_{|\alpha|}\left(\frac{\rho a}{\tau}\right)}{I_0\left(\frac{\rho a}{\tau}\right)},$$

obtenue par Ito-McKean ([6], p. 271), nous a suggéré la formule (4.9).

Rappelons que, à l'aide de cette formule «intégrée», Ito et McKean déduisent très rapidement le résultat de Spitzer [17]: sous $\mathbb{P}_{z_0}$, $\dfrac{2(\theta_t-\theta_0)}{\log t}$ converge en loi vers la distribution de Cauchy de paramètre 1, lorsque $(t\to\infty)$.

Le corollaire suivant permet, en particulier, d'établir la jonction entre le théorème (4.7) et l'étude menée par J. Pitman en [15].

(4.7.2) **Corollaire.** *Soit* $d\in\mathbb{N}$, $d\geqq 2$, *et* $\mu_d=\dfrac{d}{2}-1$. *m désigne un point fixé, non nul, de* $\mathbb{R}^d$.

$(B_u)_{u\geqq 0}$ *est un mouvement brownien, à valeurs dans* $\mathbb{R}^d$, *issu de* 0. *Alors, pour tout* $t>0$, *et* $v\in\mathbb{R}$, *on a:*

$$(*)\qquad E\left[\exp-\frac{v^2}{2}\int_t^\infty\frac{du}{\|um+B_u\|^2}\Big/\|tm+B_t\|=r\right]=\frac{I_{(\mu_d^2+v^2)^{1/2}}(r\|m\|)}{I_{\mu_d}(r\|m\|)}$$

Démonstration. On fait, dans l'intégrale entre t et ∞, le changement de variable $\mu=1/v$; puis, utilisant le fait que le processus $(vB_{1/v})_{v>0}$ est encore un mouvement brownien à valeurs dans $\mathbb{R}^d$, tendant vers 0, lorsque $v\to 0$, on applique la formule (4.8) avec $\mu=\mu_d$, $\tau=1/t$, $\rho=\dfrac{r}{t}$, $a=\|m\|$.

(4.7.3) *Remarque.* Si l'on fait, dans l'intégrale figurant en (*), le changement de variable $u=t+v$, et que l'on utilise le fait que $(B_{t+v}-B_t,\ v\geqq 0)$ est un mouvement brownien à valeurs dans $\mathbb{R}^d$, issu de 0, on peut énoncer, après introduction d'une variable gaussienne $G_{(mt,t)}$, à valeurs dans $\mathbb{R}^d$, de moyenne (mt) et de matrice de covariance $t\,\mathrm{Id}_{\mathbb{R}^d}$, indépendante du mouvement brownien d-dimensionnel $(B_t)_{t\geqq 0}$ issu de 0, la version abstraite suivante de (*):

$$\begin{aligned}&E\left[\exp\left(-\frac{v^2}{2}\int_0^\infty\frac{du}{\|G_{(mt,t)}+um+B_u\|^2}\right)\Big/\|G_{(mt,t)}\|=\rho\right]\\&=E\left[\exp\left(-\frac{v^2}{2}\int_0^\infty\frac{du}{\|uG_{(mt,t)}+m+B_u\|^2}\right)\Big/\|G_{(mt,t)}\|=\rho\right]\\&=\frac{I_{(\mu_d^2+v^2)^{1/2}}(\rho\|m\|)}{I_{\mu_d}(\rho\|m\|)}.\end{aligned}$$

(4.10) **Théorème.** *Pour tout $b>0$, on note $\tau_b=\inf\{t\geqq 0/\rho_t=b\}$. Pour tout triplet $(\mu,\nu,\beta)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+^*$, on a:*

$$E_a^\mu\left[\exp\left\{-\frac{\nu^2}{2}\int_0^{\tau_b}\frac{ds}{\rho_s^2}-\beta\tau_b\right\}\right]=\left(\frac{b}{a}\right)^\mu\frac{\mathfrak{Q}_{(\mu^2+\nu^2)^{1/2}}(a\sqrt{2\beta})}{\mathfrak{Q}_{(\mu^2+\nu^2)^{1/2}}(b\sqrt{2\beta})}, \tag{4.11}$$

où $\mathfrak{Q}=K$, si $b<a$; $\mathfrak{Q}=I$, si $a<b$.

Démonstration. Appliquons (4.6) pour le couple d'indices (μ,λ), où $\lambda=(\mu^2+\nu^2)^{1/2}$, au temps $T=\tau_b\wedge t$ $(t\geqq 0)$, et avec la variable $Y=\exp\left\{\frac{\mu^2}{2}\int_0^{\tau_b\wedge t}\frac{ds}{\rho_s^2}-\beta(\tau_b\wedge t)\right\}g(\rho_{\tau_b\wedge t})$, g désignant une fonction borélienne positive. Il vient:

$$\begin{aligned}&E_a^\mu\left[\exp\left\{-\frac{\nu^2}{2}\int_0^{\tau_b\wedge t}\frac{ds}{\rho_s^2}-\beta(\tau_b\wedge t)\right\}g(\rho_{\tau_b\wedge t})\left(\frac{\rho_{\tau_b\wedge t}}{a}\right)^{-\mu}\right]\\&=E_a^\lambda\left[\exp(-\beta(\tau_b\wedge t))\,g(\rho_{\tau_b\wedge t})\left(\frac{\rho_{\tau_b\wedge t}}{a}\right)^{-\lambda}\right]\end{aligned} \tag{4.12}$$

Il nous faut maintenant discuter selon les positions respectives de a et b.

1) Supposons $b<a$. Alors, $\rho_{\tau_b\wedge t}\geqq b$, et donc: $\left(\frac{\rho_{\tau_b\wedge t}}{a}\right)^{-1}\leqq\frac{a}{b}$. On peut donc faire tendre t vers $+\infty$ dans l'égalité (4.12)[2], ce qui donne, avec $g\equiv 1$:

$$\begin{aligned}&\left(\frac{b}{a}\right)^{-\mu}E_a^\mu\left[\exp-\left\{\frac{\nu^2}{2}\int_0^{\tau_b}\frac{ds}{\rho_s^2}+\beta\tau_b\right\}\right]\\&=\left(\frac{b}{a}\right)^{-\lambda}E_a^\lambda\left[\exp(-\beta\tau_b)\right]=\frac{K_\lambda(a\sqrt{2\beta})}{K_\lambda(b\sqrt{2\beta})},\quad\text{d'après (4.4),}\end{aligned}$$

d'où le résultat cherché.

2) Supposons $a<b$. Alors, $\rho_{\tau_b\wedge t}\leqq b$.
De même que précédemment, on peut faire tendre t vers $+\infty$ dans l'égalité (4.12), où l'on prend $g(\rho)=\rho^\lambda$. On obtient finalement (4.11). □

Le corollaire suivant découle de (3.7).

(4.10.1). **Corollaire.** *Avec les mêmes notations que dans le théorème, on a:*

$$\mathbb{E}_{z_0}\left[\exp\{i\nu(\theta_{\tau_b}-\theta_0)-\beta\tau_b\}\right]=\frac{\mathfrak{Q}_{|\nu|}(a\sqrt{2\beta})}{\mathfrak{Q}_{|\nu|}(b\sqrt{2\beta})}. \tag{4.13}$$

En faisant décroître β vers 0_+ dans l'égalité (4.11), on obtient le

(4.10.2) **Corollaire.** *Pour tout couple $(\mu,\nu)\in\mathbb{R}_+\times\mathbb{R}$, et $0<b<a<\infty$, on a:*

$$E_a^\mu\left[\exp\left\{-\frac{\nu^2}{2}\int_0^{\tau_b}\frac{ds}{\rho_s^2}\right\};\,\tau_b<\infty\right]=\left(\frac{b}{a}\right)^{\mu+(\mu^2+\nu^2)^{1/2}}.$$

---

[2] Il faut toutefois prendre garde au fait que, pour tout $\mu>0$, et $0<b<a$, on a: $P_a^\mu(\tau_b=\infty)>0$. Voir, à ce sujet, le corollaire (4.10.2). Par contre, si $b\geqq a>0$, on a, pour tout $\mu\geqq 0$, $P_a^\mu(\tau_b=\infty)=0$

En particulier, $P_a^\mu(\tau_b<\infty)=\left(\frac{b}{a}\right)^{2\mu}$.

En conclusion de ce paragraphe, remarquons que, d'après l'égalité (4.11), considérée en $\mu=0$, l'application: $(\lambda,\beta)\in\mathbb{R}_+\times\mathbb{R}_+^*\to\frac{\mathfrak{Q}_{\sqrt{2\lambda}}(a\sqrt{2\beta})}{\mathfrak{Q}_{\sqrt{2\lambda}}(b\sqrt{2\beta})}$ est la transformée de Laplace d'une mesure de probabilité sur $\mathbb{R}_+\times\mathbb{R}_+$, ce qui étend à la fois les résultats (1.4) de P. Hartman, et (4.4) de J. Kent.

Enfin, l'énoncé suivant permet d'introduire les principales notations du paragraphe 5. C'est un corollaire du théorème (4.7) (appliqué avec $\mu=0$), modulo un argument de continuité, et de (3.7).

(4.14) **Proposition.** *Pour tout* $r>0$, *il existe deux mesures de probabilité* μ_r *et* η_r, *appelées respectivement distribution développée de von Mises (sur* $\mathbb{R}$*) et distribution de Hartman-Watson (sur* $\mathbb{R}_+$*) telles que:*

(4.15) *pour tout* $v\in\mathbb{R}$, $\frac{I_{|v|}(r)}{I_0(r)}=\int_{-\infty}^{+\infty}e^{ivx}d\mu_r(x)$,

(4.16) *pour tout* $\lambda>0$, $\frac{I_{(\sqrt{2\lambda})}(r)}{I_0(r)}=\int_0^\infty e^{-\lambda u}d\eta_r(u)$.

De plus, on a:

$$d\mu_r(x)=\left(\int_0^\infty d\eta_r(u)\,e^{-x^2/2u}\frac{1}{\sqrt{2\pi u}}\right)dx, \tag{4.17}$$

et, sous $\mathbb{P}_{z_0}$, *la loi conditionnelle (sur* $\mathbb{R}_x\times\mathbb{R}_u^+$*) du couple* $\left(\theta_\tau-\theta_0;\int_0^\tau\frac{ds}{\rho_s^2}\right)$ *quand* $\rho_\tau=\rho$ *est égale à:*

$$\frac{1}{\sqrt{2\pi u}}e^{-x^2/2u}d\eta_r(u)\,dx,\quad \text{où } r=\frac{\rho a}{\tau}. \tag{4.18}$$

5. Etude des densités respectives de μ_r et η_r

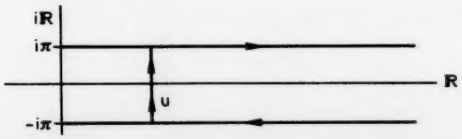
Soit $r>0$, fixé dans tout ce paragraphe, dont l'objet est d'obtenir une expression aussi explicite que possible de μ_r et η_r.

On se sert de façon intensive de la représentation intégrale suivante (cf. Petiau [14], p. 123) de $I_\lambda(r)$ $(r>0;\ \lambda\in\mathbb{R})$:

$$I_\lambda(r)=\frac{1}{2i\pi}\int_{C_u}e^{(r\operatorname{ch}\omega-\lambda\omega)}d\omega, \tag{5.1}$$

où C_u $(u\in\mathbb{R})$ désigne le chemin indiqué par la

Fig. 1



(La fonction: $\omega \to e^{r\mathrm{ch}\omega - \lambda\omega}$ étant entière, l'intégrale qui figure en (5.1) ne dépend évidemment pas de u).

En intégrant le long des deux horizontales, et de la verticale qui constituent C_0, on obtient:

$$I_\lambda(r) = \frac{1}{\pi}\int_0^\pi e^{r\cos\theta}\cos(\lambda\theta)\,d\theta - \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty e^{-r\mathrm{ch}t - \lambda t}\,dt. \tag{5.2}$$

Il nous faut encore introduire, afin d'expliciter μ_r, la fonction Φ_r: $\mathbb{R}\setminus\{0\} \to \mathbb{R}$, définie par:

$$\Phi_r(x) = \frac{x}{\pi}\int_0^\infty e^{-r\mathrm{ch}t}\frac{dt}{t^2+x^2} \quad (x\in\mathbb{R}\setminus\{0\}), \tag{5.3}$$

$$= \frac{1}{\pi}\int_0^\infty e^{-r\mathrm{ch}(xw)}\frac{dw}{1+w^2} \quad (x>0). \tag{5.3}_+$$

On déduit immédiatement de $(5.3)_+$ que $\Phi_r(0_+) = \frac{e^{-r}}{2}$, ce qui permet de prolonger Φ_r en une fonction continue à droite sur tout $\mathbb{R}$, en posant $\Phi_r(0) = \frac{e^{-r}}{2}$.

Introduisons encore les mesures:

$m(d\theta) = \frac{1}{2\pi}1_{[-\pi,\pi[}(\theta)\,d\theta$; $\pi_r(dt) = e^{-r\mathrm{ch}t}\,r\,\mathrm{sh}t\,dt$ (π_r est une mesure positive, définie *sur* $\mathbb{R}_+$, de masse totale e^{-r}), et enfin $c_t(dx) = \frac{t\,dx}{\pi(t^2+x^2)}$ $(t>0)$ la mesure de Cauchy de paramètre t.

On peut maintenant énoncer le:

(5.4) **Théorème.** *Soit* $r>0$.

(i) *μ_r se décompose en la somme $p_r + q_r$, où:*

(5.5) *$p_r(d\theta) = \frac{1}{2\pi I_0(r)}e^{r\cos\theta}1_{[-\pi,\pi[}(\theta)\,d\theta$ est la distribution de Von Mises de paramètre r.*

$$q_r(d\theta) = \frac{1}{I_0(r)}\left\{-e^{-r}m + m * \int_0^\infty \pi_r(dt)\,c_t\right\}, \tag{5.6}$$

$$= \frac{1}{2\pi I_0(r)}\{\Phi_r(\theta-\pi) - \Phi_r(\theta+\pi)\}\,d\theta \tag{5.6'}$$

est une mesure bornée, de masse totale nulle.

De plus, si l'on note f_r la densité de μ_r, définie par:

$$f_r(\theta) = \frac{1}{2\pi I_0(r)}e^{r\cos\theta}1_{[-\pi,\pi[}(\theta) + \frac{1}{2\pi I_0(r)}\{\Phi_r(\theta-\pi) - \Phi_r(\theta+\pi)\}, \tag{5.7}$$

la fonction f: $(\mathbb{R}_+\setminus\{0\})\times\mathbb{R} \to \mathbb{R}_+$ est continue.

$$(r,\theta) \to f_r(\theta)$$

(ii) $\eta_r(du)=\dfrac{1}{\pi^{3/2}(2u)^{1/2}}e^{\pi^2/2u}\dfrac{r}{I_0(r)}\psi_r(u)\,du$, *avec*

$$\psi_r(u)=\int_0^\infty dx\,e^{-x^2/2u}e^{-r\mathrm{ch}x}(\mathrm{sh}x)\sin\left(\frac{\pi x}{u}\right) \tag{5.8}$$

La fonction ψ ainsi définie sur $(\mathbb{R}_+\setminus\{0\})^2$ est continue.

Démonstration. (i) Remplaçons, en (5.2), λ par $|\nu|$ ($\nu\in\mathbb{R}$). On a:

$$\frac{1}{\pi}\int_0^\pi e^{r\cos\theta}\cos(\nu\theta)\,d\theta=\frac{1}{\pi}\int_0^\pi e^{r\cos\theta}<\frac{\varepsilon_\theta+\varepsilon_{-\theta}}{2};\ e^{i\nu\cdot}>d\theta.$$

L'application: $\nu\to\dfrac{1}{\pi I_0(r)}\displaystyle\int_0^\pi e^{r\cos\theta}\cos(\nu\theta)\mathrm{d}\theta$ apparaît donc comme la transformée de Fourier de la probabilité p_r donnée par (5.5).

D'autre part, écrivons, pour $\nu\neq 0$:

$$-\frac{\sin(|\nu|\pi)}{\pi}\int_0^\infty e^{-r\mathrm{ch}t}e^{-|\nu|t}\,dt=\left[\frac{\sin(\nu\pi)}{\nu\pi}\right](-1)\int_0^\infty e^{-r\mathrm{ch}t}\{e^{-|\nu|t}|\nu|\}\,dt$$

expression qui se transforme, par intégration par parties, en:

$$\frac{\sin(\nu\pi)}{\nu\pi}\left[-e^{-r}+\int_0^\infty\pi_r(dt)e^{-|\nu|t}\right].$$

La fonction: $\nu\to\dfrac{\sin(\nu\pi)}{\nu\pi}$ (prolongée par 1 en $\nu=0$), resp: $\nu\to e^{-|\nu|t}(t>0)$ est la transformée de Fourier de m, resp: c_t.

L'expression obtenue précédemment est donc, comme fonction de ν, la transformée de Fourier de

$$\lambda\equiv -e^{-r}m+m*\int_0^\infty\pi_r(dt)c_t,$$

d'où la formule (5.6).

La mesure λ a pour densité:

$$-\frac{e^{-r}}{2\pi}1_{[-\pi,+\pi[}(\theta)+\frac{1}{2\pi}\int_{-\pi}^{+\pi}dx\int_0^\infty\pi_r(dt)\frac{t}{\pi[t^2+(\theta-x)^2]}.$$

Lorsque l'on fait le changement de variable $t=|\theta-x|\,y$ (en t) dans l'intégrale double, celle-ci se transforme en:

$$\begin{aligned}L(\theta)&\overset{\mathrm{def}}{=}\int_{-\pi}^{+\pi}dx\int_0^\infty dy\,e^{-r\mathrm{ch}[|\theta-x|y]}r\,\mathrm{sh}[|\theta-x|y]\frac{y}{\pi(1+y^2)}\\&=\int_0^\infty\frac{dy}{\pi(1+y^2)}\int_{-\pi}^{+\pi}dx\,\mathrm{sgn}(\theta-x)e^{-r\mathrm{ch}[(\theta-x)y]}r\,\mathrm{sh}[(\theta-x)y]\,y.\end{aligned}$$

Discutons maintenant les cas: $|\theta|>\pi$, et $|\theta|<\pi$;

a) $\boxed{\theta>\pi}$:

$$L(\theta)=\int_0^\infty \frac{dy}{\pi(1+y^2)}\{e^{-r\,ch[(\theta-\pi)y]}-e^{-r\,ch[(\theta+\pi)y]}\}$$
$$=\Phi_r(\theta-\pi)-\Phi_r(\theta+\pi), \quad \text{d'après la formule } (5.3)_+ .$$

b) $\boxed{\theta<-\pi}$: Pour tout $x\in]-\pi,+\pi[$, $\operatorname{sgn}(\theta-x)=-1$.

La formule $(5.3)_+$ assure encore que $L(\theta)=\Phi_r(\theta-\pi)-\Phi_r(\theta+\pi)$.

c) $\boxed{|\theta|<\pi}$:

$$L(\theta)=\int_0^\infty \frac{dy}{\pi(1+y^2)}\left[\int_{-\pi}^{\theta} dx\, e^{-r\,ch[(\theta-x)y]}\,rsh[(\theta-x)y]\,y\right.$$
$$\left.-\int_{\theta}^{\pi} dx\, e^{-r\,ch[(\theta-x)y]}\,rsh[(\theta-x)y]y\right]$$
$$=\int_0^\infty \frac{dy}{\pi(1+y^2)}[e^{-r}-e^{-r\,ch[(\theta+\pi)y]}-e^{-r\,ch[(\theta-\pi)y]}+e^{-r}]$$
$$=e^{-r}+\Phi_r(\theta-\pi)-\Phi_r(\theta+\pi),$$

d'où la formule (5.6′) en toute généralité.

La continuité de f, définie par (5.7), se vérifie aisément (les seuls points de $\mathbb{R}$ où la continuité en θ n'est pas évidente a priori sont $-\pi$ et $+\pi$, mais le choix que nous avons fait pour $\Phi_r(0)$ entraîne le résultat cherché).

(ii) Rappelons la formule: pour tout $x>0$, et $\lambda\geqq 0$,

$$e^{-(\sqrt{2\lambda})x}=\int_0^\infty e^{-\lambda u}\frac{1}{(2\pi u^3)^{1/2}}\,x\,e^{-x^2/2u}\,du. \tag{5.10}$$

Cette formule est classique: elle exprime la transformée de Laplace du temps d'arrêt $T_x=\inf\{t/\beta_t>x\}$, où $(\beta_t)_{t\geq 0}$ désigne le mouvement brownien réel issu de 0.

D'autre part, pour tout $\varepsilon>0$, sur l'ouvert $\mathbf{0}_\varepsilon=\{z=x+iy/x^2-y^2>\varepsilon^2, x>\varepsilon\}$, l'application: $z\to\int_0^\infty e^{-\lambda u}\frac{1}{(2\pi u^3)^{1/2}}ze^{-z^2/2u}$ du est holomorphe, et l'égalité (5.10) est donc encore vraie lorsque l'on remplace x par $z\in\mathbf{0}_\varepsilon$. On peut donc écrire, d'après (5.1):

$$I_{(\sqrt{2\lambda})}(r)=\frac{1}{2i\pi}\int_{C_{\pi+\varepsilon}} d\omega\, e^{r\,ch\omega}\int_0^\infty e^{-\lambda u}\frac{1}{(2\pi u^3)^{1/2}}\omega e^{-\omega^2/2u}\,du$$
$$=\int_0^\infty e^{-\lambda u}\frac{du}{(2\pi u^3)^{1/2}}\left[\frac{1}{2i\pi}\int_{C_{\pi+\varepsilon}} e^{r\,ch\omega}\,\omega\, e^{-\omega^2/2u}\,d\omega\right],$$

l'application du théorème de Fubini étant licite.

La mesure $\eta_r(du)$ est donc absolument continue par rapport à la mesure de Lebesgue, et admet pour densité: $\frac{1}{I_0(r)}\frac{1}{(2\pi u^3)^{1/2}}H(u)$, où

$$H(u)\overset{\text{def}}{=}\frac{1}{2i\pi}\int_{C_{\pi+\varepsilon}} d\omega\, e^{r\mathrm{ch}\omega}\,\omega e^{-\omega^2/2u}=\frac{1}{2i\pi}\int_{C_0} d\omega\, e^{r\mathrm{ch}\omega}\,\omega e^{-\omega^2/2u},$$

la fonction intégrée étant entière.

En intégrant sur les deux horizontales et la verticale qui constituent C_0, il vient:

$$\begin{aligned}H(u)&=\frac{1}{2i\pi}\int_0^\infty dx\, e^{-r\mathrm{ch}x}\left[(x+i\pi)e^{-\frac{(x+i\pi)^2}{2u}}-(x-i\pi)e^{-\frac{(x-i\pi)^2}{2u}}\right]\\&=\frac{1}{2\pi}\int_0^\infty dx\, e^{-r\mathrm{ch}x}\,e^{-\frac{1}{2u}(x^2-\pi^2)}\left[-2x\sin\left(\frac{\pi x}{u}\right)+2\pi\cos\left(\frac{\pi x}{u}\right)\right]\\&=\frac{ru}{\pi}e^{\pi^2/2u}\int_0^\infty dx\, e^{-x^2/2u}\,e^{-r\mathrm{ch}x}(\mathrm{sh}x)\sin\left(\frac{\pi x}{u}\right),\end{aligned}$$

par intégration par parties. On a finalement obtenu la partie (ii) du théorème, la continuité de ψ sur $(\mathbb{R}_+\setminus\{0\})^2$ étant évidente. □

Remarque importante. On peut également démontrer la partie (i) du théorème en remarquant, à l'aide de la formule (5.1), écrite en $u>0$, que $I_{|\cdot|}(r)\in L^1(dv)$. On en déduit, par inversion de la transformation de Fourier, que:

$$\begin{aligned}f_r(\theta)&=\frac{1}{2i\pi I_0(r)}\int_{C_u} d\omega\, e^{r\mathrm{ch}\omega}\,\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iv\theta}e^{-|v|\omega}\,dv\\&=\frac{1}{2i\pi I_0(r)}\int_{C_u} d\omega\, e^{r\mathrm{ch}\omega}\,\frac{\omega}{\pi(\theta^2+\omega^2)}.\end{aligned}\tag{5.11}$$

(Il découle de cette formule que f est indéfiniment différentiable sur $(\mathbb{R}_+\setminus\{0\})\times\mathbb{R}$).

Si $|\theta|>\pi$, on peut prendre $u=0$, puis constater, après quelques calculs, que l'on a obtenu la formule (5.7).

Si $|\theta|<\pi$, on remplace C_u par le domaine d'intégration indiqué par la

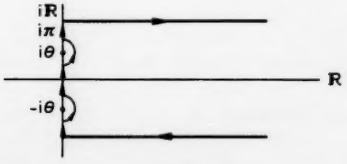


Fig. 2

ce qui, à nouveau, permet de retrouver (5.7).

Nous préférons à cette méthode la première démonstration que nous avons donnée de (5.7), car elle fait apparaître diverses probabilités (cf.: la formule (5.6)) qui ont peut-être une interprétation dans l'étude du lacet brownien. (Voir déjà le corollaire suivant).

Notons encore une formule analogue à (5.11) pour la densité $g_r(t)$ de $d\eta_r(t)$ par rapport à dt (sur $\mathbb{R}_+$), formule qui a été dégagée au cours de la démonstration précédente:

$$(5.11') \qquad g_r(t)=\frac{1}{2i\pi I_0(r)}\int_{C_u} d\omega\, e^{r\mathrm{ch}\omega}\left[\frac{\omega e^{-\omega^2/2t}}{(2\pi t^3)^{1/2}}\right]$$

Ajoutons enfin quelques remarques complémentaires sur la fonction: $x>0\to\Phi_r(x)$

a) il est immédiat, à partir de $(5.3)_+$, que:

$$(5.12) \qquad \Phi_r(x)=\frac{1}{\pi}\int_0^{\pi/2} e^{-r\mathrm{ch}(x\mathrm{tg}y)}dy \qquad (x>0).$$

b) Il est plus important de remarquer que la transformée de Fourier de: $t\to\frac{1}{2}e^{-r\mathrm{ch}t}$ est: $\xi\to K_{(i\xi)}(r)$, et celle de: $t\to\dfrac{x}{\pi(t^2+x^2)}$ $(x>0)$ est: $\xi\to e^{-|\xi|x}$.

On obtient donc, en conséquence du théorème de Plancherel[3], et de (5.3):

$$(5.13) \qquad \Phi_r(x)=\frac{1}{\pi}\int_0^{\infty} K_{(i\xi)}(r)\,e^{-\xi x}\,d\xi.^{4}$$

c) La formule (5.13) permet d'obtenir, grâce à la méthode de Laplace, l'équivalence suivante, lorsque $x\uparrow+\infty$:

si $u<v$, $\Phi_r(x+u)-\Phi_r(x+v)\simeq\dfrac{1}{\pi}K_0(r)\,(v-u)\,x^{-2}$; en particulier, si $v=-u=\pi$, on a:

$$(5.14) \qquad \Phi_r(x-\pi)-\Phi_r(x+\pi)\underset{|x|\to\infty}{\simeq} 2K_0(r)\,x^{-2}.$$

Remarquons, pour être complet, que l'on a immédiatement, à partir de (5.13):

$$\Phi_r(x)\simeq\frac{1}{\pi}K_0(r)\,x^{-1} \qquad (|x|\to\infty).$$

6. Loi de l'indice du lacet brownien

Le lemme (6.1) permettra de ramener le calcul de la loi conditionnelle de $(\theta_\tau-\theta_0)$, quand $Z_\tau=z$, sous $\mathbb{P}_{z_0}$, à celle de $(\theta_\tau-\theta_0)$, quand $\rho_\tau=\rho$ (cf., corollaire (4.7.1)).

On emploie indifféremment la notation complexe $z=x+iy$, ou bi-dimensionnelle (x, y); si $z\in\mathbb{C}\setminus\{0\}$, θ_z désigne la détermination de l'argument de z appartenant à $[-\pi, +\pi[$; si $z, z'\in\mathbb{C}$, on note $\langle z, z'\rangle=xx'+yy'$ le produit scalaire usuel de z et z'.

---

[3] Je remercie M. Barrucand qui m'a fait cette remarque

[4] On obtient une représentation du même type pour la fonction ψ_r (cf., formule (5.8)). Les calculs correspondants - inutiles ici - seront détaillés dans une Note séparée.

Soulignons que les hypothèses faites dans l'énoncé qui suit ne sont pas minimales, mais elles suffisent largement à nos besoins.

(6.1) **Lemme.** *Soient* $\mathbf{R}$ *et* θ *deux v.a. définies sur le même espace de probabilité, respectivement à valeurs dans* $\mathbb{R}_+$, *et* $\mathbb{R}$.

Soient U: $\mathbb{R}_+ \to \mathbb{R}_+$, *et* V: $\begin{matrix}\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \\ (\rho, u) \to V_\rho(u)\end{matrix}$ *deux fonctions boréliennes telles que:*

$$dP_{\mathbf{R}}(\rho) = d\rho\, \rho\, U(\rho) \quad \text{et} \quad dP_{\theta/\mathbf{R}=\rho}(\theta) = d\theta\, V_\rho(\theta).$$

Alors, si l'on note $\mathbf{Z} = \mathbf{R}e^{i\theta}$, *on a*:

(i) $dP_{\mathbf{Z}}(x, y) = dx\, dy\, p(z)$, où $p(z) \equiv \sum_{n\in\mathbb{Z}} U(|z|)\, V_{|z|}(\theta_z + 2n\pi)$ $(z \neq 0)$.

(ii) $dP_{\theta/\mathbf{Z}=z} = \sum_{n\in\mathbb{Z}} \dfrac{U(|z|)\, V_{|z|}(\theta_z + 2n\pi)}{p(z)}\, \varepsilon_{(\theta_z + 2n\pi)}$.

(iii) *Si, de plus, les applications* U, V *et* p *(supposée identiquement finie, et strictement positive) sont continues, respectivement sur* $\mathbb{R}_+^*$, $\mathbb{R}_+^* \times \mathbb{R}$, *et* $\mathbb{C}\setminus\{0\}$, *alors, pour toute fonction continue, bornée et à support compact* ϕ: $\mathbb{R} \to \mathbb{R}$, *l'application*

$$\square_\phi : z \to \sum_{n\in\mathbb{Z}} \frac{U(|z|)\, V_{|z|}(\theta_z + 2n\pi)}{p(z)}\, \phi(\theta_z + 2n\pi)$$

est continue sur $\mathbb{C}\setminus\{0\}$.

Démonstration. (i) et (ii) découlent simplement de la décomposition de l'intégrale (générique!) en $d\theta$, soit $\int_{-\infty}^{+\infty} d\theta[\ldots]$ en $\sum_{n\in\mathbb{Z}} \int_{2n\pi-\pi}^{2n\pi+\pi} d\theta[\ldots]$, et du changement de variable $\theta = 2n\pi + u$ dans l'intégrale correspondante.

Pour démontrer (iii), on commence par remarquer que l'on peut remplacer – à l'évidence – dans la définition des fonctions p et $\square_\phi$ la détermination principale θ_z de l'argument de $z \in \mathbb{C}\setminus\{0\}$ par n'importe quelle détermination.

L'assertion (iii) est alors une conséquence immédiate de l'existence, au voisinage de tout point $u \in \mathbb{C}\setminus\{0\}$, d'une détermination continue de l'argument.

Remarque. Si U et V sont continues, respectivement sur $\mathbb{R}_+^*$, et $\mathbb{R}_+^* \times \mathbb{R}$, et si $P_{\mathbf{Z}}(dx \times dy)$ admet une densité p_0 continue (*finie*) par rapport à $dx\,dy$, alors la fonction p du lemme (6.1) est identiquement finie sur $\mathbb{C}\setminus\{0\}$ (et égale $dx\,dy$ p.s. à p_0).

Rappelons maintenant que, selon les notations du paragraphe 2, $\mathbb{P}_{z_0}^z$ $(zz_0 \neq 0)$ désigne la loi du pont brownien complexe $PB(z_0, z)$ (entre 0 et $\tau > 0$); on avait pris, pour simplifier, $\tau = 1$ dans le paragraphe 2.

(6.2) **Théorème.** *Soit* $z_1 \neq 0$. *On note* $r = \dfrac{|z_1 z_0|}{\tau}$ *et* $\theta_{(0,1)} = \theta_{z_1 z_0^{-1}}$. *Pour toute*

fonction h: $\mathbb{R}\to\mathbb{R}$ *borélienne, bornée, on a*:

$$\text{(6.3)}\quad \mathbb{E}^{z_1}_{z_0}(h(\theta_\tau-\theta_0))=e^{-\frac{\langle z_1,z_0\rangle}{\tau}}\{2\pi I_0(r)\}\sum_{n\in\mathbb{Z}} f_r(\theta_{(0,1)}+2n\pi)\,h(\theta_{(0,1)}+2n\pi)^5$$

$$\text{(6.4)}\quad =h(\theta_{(0,1)})+e^{-\frac{\langle z_1,z_0\rangle}{\tau}}\sum_{n\in\mathbb{Z}}[\Phi_r(\theta_{(0,1)}+(2n-1)\pi)-\Phi_r(\theta_{(0,1)}$$

$$+(2n+1)\pi)]\,h(\theta_{(0,1)}+2n\pi)$$

Démonstration. Posons $u_0=\dfrac{z_0}{|z_0|}$, $z_1=zu_0$, et $a=|z_0|$. Avec ces nouvelles notations, l'égalité (6.3) se transforme en:

$$\text{(6.3')}\quad \mathbb{E}^{zu_0}_{z_0}(h(\theta_\tau-\theta_0))=\sum_{n\in\mathbb{Z}} 2\pi I_0(r)e^{-\frac{ax}{\tau}} f_r(\theta_z+2n\pi)\,h(\theta_z+2n\pi).$$

Il suffit, pour démontrer (6.3'), d'appliquer le lemme (6.1) avec $\mathbf{Z}=Z_\tau u_0^{-1}$ et $\boldsymbol{\theta}=\theta_\tau-\theta_0$. Remarquons que $\mathbf{R}=\rho_\tau$, et que, avec les notations du lemme

a) $U(\rho)=\dfrac{1}{\tau}e^{-\frac{\rho^2+a^2}{2\tau}}I_0\left(\dfrac{\rho a}{\tau}\right)$ (d'après (4.3), ou par un calcul direct).

b) $V_\rho(\theta)=f_\rho(\theta)$.

c) $p(z)=\dfrac{1}{2\pi\tau}e^{-\frac{1}{2}\left[\frac{|z|^2+a^2}{\tau}\right]}e^{\frac{ax}{\tau}}$

(Cette identité découle, de façon simple, du lemme (6.1), de la remarque qui le suit, et de la formule (5.7) explicitant f.)

Les hypothèses du lemme (6.1) (partie (iii)) sont bien vérifiées grâce au théorème (5.4). Ainsi, si h est une fonction continue, bornée, à support compact, les deux membres de (6.3') sont des fonctions continues de $z(\in\mathbb{C}\setminus\{0\})$ (le membre de gauche l'est d'après le lemme (2.12)), qui sont égales dz p.s., et donc identiques. L'égalité (6.4) découle de (6.3), et de la formule (5.7). □

Le résultat suivant est une conséquence immédiate de (5.14).

(6.2.1) **Corollaire.** *Avec les notations du théorème* (6.2), *on a:*

$$\mathbb{P}^{z_1}_{z_0}(\theta_\tau-\theta_0=\theta_{(0,1)}+2n\pi)\underset{|n|\to\infty}{\simeq}\frac{K_0(r)}{2\pi^2n^2}e^{-\frac{\langle z_1,z_0\rangle}{\tau}}.$$

Le théorème ci-dessous étend les résultats du théorème (6.2).

(6.5) **Théorème.** *Avec les notations du théorème précédent, on a, pour tout* $z_1\neq 0$, *et toute fonction* H: $\mathbb{R}\times\mathbb{R}_+\to\mathbb{C}$, *borélienne, bornée:*

$$\text{(6.6)}\quad \mathbb{E}^{z_1}_{z_0}\left(H\left(\theta_\tau-\theta_0;\int_0^\tau\frac{ds}{\rho_s^2}\right)\right)^6$$

$$=I_0(r)e^{-\frac{\langle z_1,z_0\rangle}{\tau}}\int_0^\infty d\eta_r(u)\sqrt{\frac{2\pi}{u}}\left[\sum_{n\in\mathbb{Z}}e^{-\frac{(2n\pi+\theta_{(0,1)})^2}{2u}}H(\theta_{(0,1)}+2n\pi;u)\right]$$

---

[5] cf., la formule (5.7) où est définie la fonction f

[6] Rappelons que la mesure η_r a été définie au moyen de la formule (4.16), et explicitée dans le théorème (5.4), ii)

Plus particulièrement, pour toute fonction $K\colon \mathbb{R}_+ \to \mathbb{C}$, *borélienne, bornée, et tout* $\lambda \in \mathbb{R}$, *on a:*

$$\mathbb{E}_{z_0}^{z_1}\left(e^{i\lambda(\theta_\tau-\theta_0)}K\left(\int_0^\tau \frac{ds}{\rho_s^2}\right)\right) = I_0(r)\,e^{-\frac{\langle z_1,z_0\rangle}{\tau}}\int_0^\infty d\eta_r(u)\Big[\sum_{n\in\mathbb{Z}} e^{-\left[\frac{(n+\lambda)^2}{2}+ni\theta_{(0,1)}\right]}\Big]K(u), \tag{6.7}$$

et enfin, on a, pour tous λ, $\nu \in \mathbb{R}$:

$$\mathbb{E}_{z_0}^{z_1}(e^{i\lambda(\theta_\tau-\theta_0)}e^{-\nu^2/2\int_0^\tau ds/\rho_s^2}) = e^{-\frac{\langle z_1,z_0\rangle}{\tau}}\sum_{n\in\mathbb{Z}} e^{-in\theta_{(0,1)}}I_{[(n+\lambda)^2+\nu^2]^{1/2}}(r) \tag{6.8}$$

Démonstration. 1) La démonstration de la formule (6.6) est tout à fait semblable à celle de (6.3), la seule différence consistant à faire usage maintenant de la formule (4.18).

2) La formule (6.7) découle de (6.6), où l'on prend $H(\theta,u)=e^{i\lambda\theta}K(u)$, via l'égalité (on note simplement ici θ pour $\theta_{(0,1)}$):

$$\sqrt{\frac{2\pi}{u}}\Big[\sum_{n\in\mathbb{Z}} e^{-\frac{(2n\pi+\theta)^2}{2u}}e^{i\lambda(\theta+2n\pi)}\Big] = \sum_{n\in\mathbb{Z}} e^{-\left[\frac{(n+\lambda)^2u}{2}+ni\theta\right]},$$

elle même conséquence immédiate de la «formule de Poisson»: (cf, par exemple, [18], théorème 5, p. 153)

$$\sum_{n\in\mathbb{Z}} e^{-\pi(n+z)^2a} = \frac{1}{\sqrt{a}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/a+2i\pi nz}$$

pour tout $a>0$, et tout $z\in\mathbb{C}$.

3) Enfin, remarquons que:

$$\int_0^\infty d\eta_r(u)\Big[\sum_{n\in\mathbb{Z}} e^{-\frac{(n+\lambda)^2u}{2}}\Big] < \infty,$$

d'après la formule (6.7), prise en $\theta_{(0,1)}=0$. On a donc, d'après la même formule:

$$\mathbb{E}_{z_0}^{z_1}(e^{i\lambda(\theta_\tau-\theta_0)}e^{-\nu^2/2\int_0^\tau ds/\rho_s^2}) = I_0(r)\,e^{-\frac{\langle z_1,z_0\rangle}{\tau}}\sum_{n\in\mathbb{Z}} e^{-in\theta_{(0,1)}}\left\{\frac{I_{[(n+\lambda)^2+\nu^2]^{1/2}}(r)}{I_0(r)}\right\}$$

d'après la définition de η_r (formule (4.16)). □

Avant d'étudier le cas particulièrement intéressant où $z_0=z_1$, donnons une autre version de certains des résultats du théorème (6.5): soit $d\geqq 2$, et m, m' deux points non nuls de $\mathbb{R}^d$; si $(B_t)_{t\geq 0}$ est un mouvement brownien à valeurs dans $\mathbb{R}^d$, issu de 0, la distribution $\lambda_{(m;m')}$ de $\int_0^\infty \frac{du}{\|m+m'u+B_u\|^2}$ est symétrique en m et m',

c'est à dire: $\lambda_{(m,m')}=\lambda_{(m',m)}$, ce qui se démontre, comme pour le corollaire (4.7.2), en utilisant le fait que $(vB_{1/v})_{v>0}$ est encore un mouvement brownien. Pour $d=2$, la même propriété, appliquée au calcul de la loi conditionnelle de $\int_1^\infty \frac{du}{\|mu+B_u\|^2}$ quand $m+B_1=m'$, nous donne, d'après la formule (6.7): pour toute fonction K: $\mathbb{R}_+\to\mathbb{C}$, borélienne, bornée,

$$\begin{aligned}(6.9)\qquad &\int K(x)\,d\lambda_{(m,m')}(x)\\ &\quad=I_0(r)e^{-\langle m,m'\rangle}\int_0^\infty d\eta_r(u)\Big[\sum_{n\in\mathbb{Z}} e^{-\left[\frac{n^2u}{2}+i\theta_{(0,1)}n\right]}\Big]K(u)\end{aligned}$$

où $r=\|m\|\,\|m'\|$, et $\theta_{(0,1)}=\theta_{m'm^{-1}}$.

Les théorèmes (6.2) et (6.5) permettent en particulier de calculer la loi du couple $\left(N_\tau;\int_0^\tau \frac{ds}{\rho_s^2}\right)$, sous $\mathbb{P}^*_{z_0}\overset{\text{def}}{=}\mathbb{P}^{z_0}_{z_0}$, où l'on note $N_\tau=\frac{1}{2\pi}(\theta_\tau-\theta_0)$ l'indice de 0 par rapport au lacet brownien $(Z_u(\omega),0\leqq u\leqq\tau)$. On note dorénavant $r=\frac{|z_0|^2}{\tau}$.

(6.10) **Théorème.** 1) *Sous $\mathbb{P}^*_{z_0}$, la loi de l'indice (N_τ) est:*

$$(6.4^*)\qquad \varepsilon_0(dm)+e^{-r}\sum_{n\in\mathbb{Z}}[\Phi_r((2n-1)\pi)-\Phi_r((2n+1)\pi)]\,\varepsilon_n(dm)$$

2) *Sous $\mathbb{P}^*_{z_0}$, la loi du couple $\left(N_\tau;\int_0^\tau \frac{ds}{\rho_s^2}\right)$ est:*

$$(6.6^*)\qquad I_0(r)e^{-r}\,d\eta_r(u)\sqrt{\frac{2\pi}{u}}\Big[\sum_{n\in\mathbb{Z}} e^{-\frac{2n^2\pi^2}{u}}\,\varepsilon_n(dm)\Big].$$

Pour conclure, soulignons que la fonction caractéristique de N_τ, sous $\mathbb{P}^*_{z_0}$ a la forme particulièrement simple suivante:

$$(6.8^*)\qquad \mathbb{E}^*_{z_0}(e^{2i\pi\lambda N_\tau})=e^{-r}\sum_{n\in\mathbb{Z}} I_{|n+\lambda|}(r)$$

7. Extension de certains résultats en dimension supérieure à 2

Dans tout ce paragraphe, d désigne un entier supérieur à 2, et pour tout $m\in\mathbb{R}^d$, $\mathbb{P}_m$ est la loi - définie, comme d'habitude, sur l'espace canonique $C(\mathbb{R}_+,\mathbb{R}^d)$ - du mouvement brownien à valeurs dans $\mathbb{R}^d$, issu de m en $t=0$.

$\tau>0$ étant fixé, le but du paragraphe est de donner, de façon aussi explicite que possible, la loi de $\int_0^\tau ds/\|B_s\|^2$, sous $\mathbb{P}_m$, conditionnellement à $(B_\tau=m')$ $(m,m'\neq 0$; on note encore $\mathbb{P}_m^{m'}$ la loi du pont brownien ainsi défini entre 0 et τ).

Rappelons, en préliminaire, les résultats classiques suivants:

(7.1) Le processus (B_t) peut se représenter comme «skew-product» de sa partie radiale $\rho_t=\|B_t\|$, et de sa partie «angulaire» $\sigma_t\equiv\theta_{\left(\int_0^t ds/\rho_s^2\right)}$, à valeurs dans la sphère

unité S_{d-1}, le processus $(\theta_t)_{t\geqq 0}$ étant un mouvement brownien standard à valeurs dans S_{d-1}, indépendant de ρ (cf. Ito-Mc Kean [6], p. 270)

(7.2) La densité, au temps t, du processus $(\theta_t)_{t\geqq 0}$ à valeurs dans S_{d-1}, issu de θ en $t=0$, par rapport à la mesure de surface $d\sigma(\phi)$ sur S_{d-1} est donnée par:

$$p_t^{(d)}(\theta,\phi)=\sum_{n=0}^{\infty} e^{-\frac{n(n+d-2)t}{2}}\frac{N(d;n)}{\omega_d}P_n(d;\langle\theta,\phi\rangle)$$

(cf, Ito-Mc Kean [6], p. 275, ou F. Perrin [19], formule (59), p. 25), où:

$N(d;n)$ est le nombre d'harmoniques sphériques linéairement indépendantes, de degré n, en dimension d.

ω_d est la surface de la sphère unité de $\mathbb{R}^d$.

$P_n(d;.)$ est le polynome de Legendre de degré n, en dimension d.

$\langle\theta,\phi\rangle$ est le produit scalaire des points θ et ϕ de $S_{d-1}\subset\mathbb{R}^d$.

On voit aisément, en conséquence de l'égalité: $P_n(d;0)=1$, que la série définissant $p^{(d)}$ est majorée, terme à terme, par:

$$e^{-\frac{n(n+d-2)t}{2}}\frac{N(d;n)}{\omega_d},$$

et est donc uniformément absolument convergente. On peut maintenant énoncer le:

(7.3) **Théorème.** *Notons* $\mu_d=\frac{d}{2}-1$. *Soient* $m,m'\in\mathbb{R}^d$, *non nuls;* $\theta=m/\|m\|$; $\theta'=m'/\|m'\|$. *On note encore* $r=\frac{\|m\|\ \|m'\|}{\tau}$. *Alors, pour toute fonction borélienne* $\phi:\mathbb{R}_+\to\mathbb{R}_+$, *on a*

$$\text{(7.4)}\qquad \mathbb{E}_m^{m'}\left[\phi\left(\int_0^\tau\frac{ds}{\rho_s^2}\right)\right]=e^{-\frac{\langle m,m'\rangle}{\tau}}(2\pi)^{d/2}\frac{I_0(r)}{r^{\mu_d}}\int_0^\infty d\eta_\tau(v)\,e^{-\mu_d^2 v/2}\,\phi(v)\,p_v^{(d)}(\theta;\theta')$$

et, en particulier, pour tout $v\in\mathbb{R}$:

$$\text{(7.5)}\qquad \begin{aligned}&\mathbb{E}_m^{m'}\left(e^{-v^2/2\int_0^\tau ds/\rho_s^2}\right)\\&=e^{-\frac{\langle m,m'\rangle}{\tau}}\frac{(2\pi)^{d/2}}{r^{\mu_d}}\left[\sum_{(n\geqq 0)}I_{[(n+\mu_d)^2+v^2]^{1/2}}(r)\frac{N(d;n)}{\omega_d}P_n(d;\langle\theta,\theta'\rangle)\right]\end{aligned}$$

Démonstration. La formule (7.5) découle immédiatement de (7.4), où l'on prend $\phi(v)=e^{-v^2v/2}$, de (7.2), et enfin de la définition de η_τ par sa transformée de Laplace (formule (4.16)).

Afin de démontrer la formule (7.4), supposons ϕ bornée, et introduisons $f:\mathbb{R}^d\to\mathbb{R}$, borélienne, bornée.

On a alors, d'après (7.1), en notant $V_\tau=\int_0^\tau ds/\rho_s^2$:

$$\begin{aligned}&\mathbb{E}_m[f(B_\tau)\,\phi(V_\tau)]\\&=\int\mathbb{P}_m(\rho_\tau\in d\rho;\,V_\tau\in dv)\,\phi(v)\int_{S_{d-1}}d\sigma(\phi)\,p_v^{(d)}(\theta;\phi)\,f(\rho\phi)\\&=\int\mathbb{P}_m(\rho_\tau\in d\rho)\int d\eta_{\frac{\rho\|m\|}{\tau}}^{(d)}(v)\,\phi(v)\int_{S_{d-1}}d\sigma(\phi)\,p_v^{(d)}(\theta;\phi)\,f(\rho\phi),\end{aligned}$$

où, d'après la formule (4.8), $\eta_r^{(d)}(dv)$ désigne la probabilité sur $\mathbb{R}_+$, dont la transformée de Laplace, au point $\left(\frac{v^2}{2}\right)$, est égale à: $\frac{I_{(\mu_d^2+v^2)^{1/2}}(r)}{I_{\mu_d}(r)}$. On a, à l'évidence: $d\eta_r^{(d)}(v)=\frac{I_0(r)}{I_{\mu_d}(r)}d\eta_r(v)\,e^{-\mu_d^2 v/2}$.

On termine la démonstration en explicitant la densité de $\mathbb{P}_m(\rho_\tau \in d\rho)$ par rapport à $d\rho$, puis en reprenant les coordonnées cartésiennes à la place du couple (ρ, ϕ). □

Si l'on prend $\phi \equiv 1$ dans la formule (7.4), on obtient le:

(7.3.1) **Corollaire** (Hartman-Watson, [4], p. 594, identité (1.5)). *Avec les mêmes notations que dans le théorème, on a:*

$$e^{\frac{\langle m,m'\rangle}{\tau}}\frac{r^{\mu_d}}{(2\pi)^{d/2}}=I_0(r)\int_0^\infty d\eta_r(v)\,e^{-\mu_d^2 v/2}\,p_v^{(d)}(\theta;\theta').$$

Soulignons ici que, d'après les arguments qui précèdent l'établissement de la formule (6.9), on a, pour $m, m' \neq 0$, et (B_t) mouvement brownien à valeurs dans R^d, issu de 0:

$$E\left(e^{-v^2/2\int_0^\infty \frac{du}{\|m+m'u+B_u\|^2}}\right)=\mathbb{E}_m^{m'}\left(e^{-v^2/2\int_0^1 ds/\rho_s^2}\right)$$

d'où l'obtention de la mesure $\lambda_{(m,m')}$ sur $\mathbb{R}_+$, pour toutes les dimensions $d \geqq 2$.

(7.6) *Remarque.* La formule (7.4) est, bien entendu, également valable pour $d=2$, et l'on retrouve alors (partiellement) les formules (6.6) et (6.7), respectivement pour H ne dépendant pas de θ, et $\lambda=0$. Le passage de (6.6) à (6.7) résulte d'ailleurs simplement de la représentation du mouvement brownien standard sur S_1 comme l'image du mouvement brownien réel par la congruence modulo 2π. Cette remarque a déjà été faite par F. Perrin ([19], p. 22). En dimension $d>2$, le même auteur donne (formules (61), p. 26) une expression de $p_t^{(d)}(\theta;\theta')$ différente de (7.2), mais qu'il semble difficile d'interpréter du point de vue probabiliste.

8. Remarques finales

a) J. Pitman étudie, en [15], certaines propriétés de la norme du mouvement brownien à valeurs dans $\mathbb{R}^d$, avec drift constant.

Après lecture d'une version préliminaire de notre travail, il a déduit la formule (*) (corollaire (4.7.2)) pour $d=2$, du théorème (4.7).

La démonstration du corollaire (4.7.2) présentée ici est postérieure à celle de Pitman, mais nous semble plus immédiate.

b) Egalement après la lecture de ce papier, J. Henry nous a appris que la loi de l'indice du lacet Brownien avait déjà été calculée par S.F. Edwards [20] par une méthode tout à fait différente de la nôtre.

c) La densité de la distribution de von Mises développée (Théorème (5.4)) μ_r a été obtenue initialement par D.G. Kendall, et figure (p. 212–219) dans la thèse de Ph.D. de J. Kent, intitulée: «Distributions, processes and statistics on spheres»

(University of Cambridge - Novembre 1976)

Toutefois, à la connaissance de Kent, ce résultat ne semble pas avoir été publié jusqu'à présent. Nous n'avons pas trouvé trace non plus, dans la littérature, du calcul de la densité de η_r (théorème (5.4) également).

Je profite de ces remarques pour remercier vivement J. Pitman, J. Henry et J. Kent de l'intérêt qu'ils ont porté à la réalisation de ce travail.

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Nonparametric Estimation Based on Censored Observations of a Markov Renewal Process

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Summary. Uniform consistency and weak convergence is proved of estimators of the transition probabilities of an arbitrary finite state space Markov renewal process, based on n independent and identically distributed "right censored" realizations of the process. The approach uses the theory of stochastic integrals and counting processes. It is shown how the results may be extended to the non-identically distributed case and to general censorship under suitable conditions.

1. Introduction

A Markov renewal process can often be used to describe the lengths of time spent in consecutive stages (not necessarily following a fixed order) of a disease, or in the functioning of a machine. In such a situation one might want to estimate the distribution functions (so called *transition probabilities*) which describe the probabilistic behaviour of this phenomenon. Lagakos et al. [9] (see this paper and its references for applications) proposed certain estimators on the basis of maximum likelihood considerations by maximizing the probability of n realizations of the process observed on fixed finite time intervals over all discrete transition probabilities; they also derived approximate variances and covariances of the resulting estimators by looking at the Fisher information, again assuming discrete distributions. The present paper gives rigorous derivations of consistency and weak convergence of these estimators as n, the number of (partial) realizations of the process, tends to infinity. We make *no* assumptions on the transition probabilities and also allow quite general types of *censoring*: that is to say, the mechanism through which only incomplete realizations of the Markov renewal process are available.

Let $J_0, J_1, \ldots$ be the consecutive states of a Markov renewal process and let $X_1, X_2, \ldots$ be the sojourn times in these states (we take our notation from Pyke [13] apart from two changes which we mention shortly). So $J_0, J_1, \ldots$ are r.v.'s taking values in the finite set of states $\{1, \ldots, m\}$ for some $m \in \mathbb{N}$ and $X_1, X_2, \ldots$

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are r.v.'s taking values in $(0, \infty]$ with the interpretation that the process starts off at time $S_0=0$ in state J_0, and at time $S_n=\sum_{i=1}^{n} X_i$, after a stay of length X_n in state J_{n-1}, jumps to state J_n (at least, if $S_n<\infty$). The joint distribution of J_0, X_1, J_1, X_2, J_2, ... is built up recursively from an initial state distribution

$$\alpha_k=P(J_0=k) \qquad k\leqq m \tag{1}$$

and transition probabilities (which we want to estimate)

$$Q_{ij}(t)=P(X_n\leqq t, J_n=j|J_{n-1}=i), \qquad i,j\leqq m,\ t\in[0,\infty],\ n\in\mathbb{N}, \tag{2}$$

satisfying $Q_{ij}(0)=0$ by supposing that for each $n\in\mathbb{N}$

$$P(X_n\leqq t, J_n=j|J_0, X_1, J_1, \ldots, X_{n-1}, J_{n-1})=P(X_n\leqq t, J_n=j|J_{n-1}) \quad \text{a.s.} \tag{3}$$

We shall also be interested in estimating the (possibly defective) distribution functions H_i, defined by

$$H_i(t)=P(X_n\leqq t|J_{n-1}=i)=\sum_j Q_{ij}(t). \tag{4}$$

Rather than working with the r.v.'s $J_0, X_1, J_1, \ldots$ we step over to *counting processes* $\tilde{N}_{ij}$ (Aalen [1]) which register the transitions made from state i to state j up to time t:

$$\tilde{N}_{ij}(t)=\#\{n\geqq 1: S_n\leqq t, J_{n-1}=i, J_n=j\} \qquad i,j\leqq m,\ t\in[0,\infty). \tag{5}$$

Because the state space is finite, by Pyke [12, Lemma 4.1] these processes have sample paths which (almost surely) are finite for all t, zero at time zero, integer valued and right continuous with jumps of size $+1$ only, these jumps furthermore not occurring simultaneously in different processes. Define also

$$\begin{aligned}
\tilde{N}_{i\cdot}(t)&=\sum_j \tilde{N}_{ij}(t)\\
\tilde{N}_{\cdot j}(t)&=\sum_i \tilde{N}_{ij}(t) \qquad \text{(denoted } N_j(t) \text{ in Pyke [12])}\\
\tilde{N}(t)&=\sum_{i,j} \tilde{N}_{ij}(t) \qquad \text{(denoted } N(t) \text{ in Pyke [12])}
\end{aligned} \tag{6}$$

and define

$$\begin{aligned}
Z(t)&=J_{\tilde{N}(t)}\\
L(t)&=t-S_{\tilde{N}(t-)}.
\end{aligned} \tag{7}$$

Pyke [12] calls the process Z, the state occupied at time $t+$, a *semi-Markov process*, and the multivariate process $\{\tilde{N}_{\cdot j}; j\leqq m\}$ a *Markov renewal process*. $L(t)$ is a left continuous version of what is called the *backward recurrence time*; it is the length of time which at time $t-$ has elapsed since the last jump of the Markov renewal process. It plays an important role in the sequel.

We shall later use the fact that $\tilde{N}(t)$ has finite moments of all orders. For, as m is finite, there exists a distribution function F such that $F(0)=0$ and

$Q_{ij}(t) \leqq F(t)$ for all i, j and t. So $\tilde{N}(t)$ is stochastically dominated by the evaluation at time t of a renewal process having recurrence time distribution F; and for this process the result is well known.

The rest of this paper is organized as follows. We shall conclude this section with some more notation and some facts from the theory of stochastic integrals and counting processes for use in Sect. 2. There we formally define our censoring model, by introducing a process K taking the values 0 and 1, with the interpretation that the Markov renewal process is "under observation" at time t if and only if $K(t)=1$. We assume that K is *predictable* which loosely speaking means that $K(t)$ does not depend on the development of the Markov renewal process in $[t, \infty)$. It is easy to construct counter examples where K does not have this property and the estimators considered here are inconsistent. Some further technical assumptions are made, such as a restriction to "right censorship". Then we derive some important equalities involving the first and second moments of processes counting sojourn times in the various states "observed" to be $\leqq t$ and to be followed by a transition to a particular state, or just "observed" to be $\geqq t$. In Sect. 3 we give definitions of the estimators of Lagakos et al. [9] based on n independent and identically distributed censored observations of the Markov renewal process. The estimators depend on the data through the processes counting observed sojourn times. We now prove three theorems: on consistency of the estimators with arbitrary Q_{ij}'s, on asymptotic normality with discrete Q_{ij}'s, and on weak convergence with arbitrary Q_{ij}'s. Theorem 2 with its easy proof (once Theorem 1 has been established) is included to illustrate Theorem 3, without the latter's technicalities (which are themselves largely postponed to Sect. 5 in the form of some lemmas). Section 4 discusses these results and shows how the assumptions of right censorship and identical distributions may be relaxed.

Notation. For a real valued stochastic process $Y=\{Y(t); t\in[0, \infty)\}$ whose sample paths have left hand limits, Y_- is the process defined by

$$Y_-(0)=0$$

and

$$Y_-(t)=Y(t-).$$

In dealing with an indexed family of processes, we write e.g. Y_{i-} for $(Y_i)_-$. We define Y_+ similarly under the obvious conditions. ΔY is the process $Y-Y_-$ and $\mathscr{E}Y$ the function $t\to\mathscr{E}Y(t)$. By $\int X\,dY$ we denote (with a single exception in Theorem 3) the process with sample paths $t\to\int_{[0,t]} X(s)\,dY(s)$, the latter being interpreted as a pathwise Lebesgue-Stieltjes integral, such an interpretation being possible in all the cases considered. We write $\int^- X\,dY$ for the process $(\int X\,dY)_-$ with sample paths $t\to\int_{[0,t)} X(s)\,dY(s)$. Some miscellaneous notations are χ_A for the indicator function of the set A and $\|\cdot\|_\tau$ for the supremum norm on $[0,\tau]$, also written $\|\cdot\|$ if there can be no confusion. Unless otherwise stated, time variables s, t, τ, etc. are always in $[0,\infty)$ and state variables i,j in $\{1,\ldots,m\}$. The σ-algebra generated by a family of r.v.'s is denoted by $\sigma\{\cdot\}$,

while to indicate that generated by a union of σ-algebras we use the symbol $\vee$. Maximum and minimum are denoted by $\vee$ and $\wedge$ respectively.

Stochastic Integrals. We collect together here a few results from Meyer [11]. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\{\mathcal{F}_t; t\in[0,\infty)\}$ be an increasing right continuous family of sub σ-algebras of Ω such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. A process X is *predictable* if as a function of (t,ω) it is measurable with respect to the σ-algebra on $[0,\infty)\times\Omega$ generated by the left continuous adapted processes. A process X possesses a property *locally* if there exists a so called *localizing* sequence of stopping times $\{T_n\}$, $T_n\uparrow\infty$, such that for each n the process $t\to\chi_{\{T_n>0\}}X(t\wedge T_n)$ has this property.

Martingales are always supposed to have right continuous paths with left hand limits. Let M and N be local square integrable martingales and let H and K be predictable and locally bounded. There exists a unique predictable process $\langle M,N\rangle$ with paths of locally integrable variation such that $MN-\langle M,N\rangle$ is a local martingale, zero at time zero. A process $H\circ M$ is uniquely defined by requiring that

(i) it should be a local square integrable martingale, and

(ii) $\langle H\circ M, H\circ M\rangle = \int H^2\, d\langle M,M\rangle$.

Almost everywhere where the path-wise Lebesgue-Stieltjes integral $\int H\,dM$ is well defined, the paths of $H\circ M$ and $\int H\,dM$ coincide. Furthermore $\langle H\circ M, K\circ N\rangle = \int HK\, d\langle M,N\rangle$. We actually make most use of the simple corollaries of these results, that if the processes $\int H\,dM$ and $\int K\,dN$ exist, if all the localizing stopping times above can be taken as constants, and if $M(0)=N(0)=0$, then the following equalities (between functions on $[0,\infty)$) hold:

$$\mathcal{E}\left(\int H\,dM\right)=\mathcal{E}\left(\int K\,dN\right)=0, \tag{8}$$

$$\mathcal{E}\left(\int H\,dM\int K\,dN\right)=\mathcal{E}\left(\int HK\,d\langle M,N\rangle\right). \tag{9}$$

If the word "local" can be dropped above, then the same relationships hold on $[0,\infty]$.

Counting Processes. Let $\{N_i; i=1,\ldots,k\}$ be an indexed family of processes with right continuous paths, which are zero at time zero, nondecreasing and integer valued, and have jumps of size $+1$ only, no two processes jumping at the same time. Suppose that for all t,

$$\mathcal{F}_t=\mathcal{F}_0\vee\sigma\{N_i(s); i=1,\ldots,k, s\leqq t\}.$$

Define $T_0=0$ and $T_n=\inf\{t\colon \sum_i N_i(t)\geqq n\}$, $n=1,2,\ldots$, i.e. T_n is the time of the n-th jump of $\{N_i;\ i=1,\ldots,k\}$. For $n\geqq 1$ let J_n be the index of the particular process which if T_n is finite jumps at time T_n, i.e. $N_{J_n}(T_n)=N_{J_n}(T_n-)+1$ if $T_n<\infty$. Define $F_{ni}(t)$ to be a regular version of $P(T_n\leqq t \text{ and } J_n=i\,|\,\mathcal{F}_{T_{n-1}})$, $t\in[0,\infty)$, $i=1,\ldots,k$, and $n=1,2,\ldots$, and

$$A_i(t)=\sum_{n=1}^{\infty}\int_0^t \frac{dF_{ni}(s)}{1-\sum_{i'=1}^{k}F_{ni'}(s-)}. \tag{10}$$

Jacod [8] shows, for $i=1,\ldots,k$, that $M_i=N_i-A_i$ is a local martingale, zero at time zero, while by Gill [6, Propositions 12 and 13], or Liptser and Shiryaev [10, Lemma 18.12], the M_i's are local square integrable martingales with

$$\langle M_i, M_j\rangle = \begin{cases} \int (1-\Delta A_i)\, dA_i & i=j \\ -\int \Delta A_i\, dA_j & i\neq j. \end{cases} \tag{11}$$

The localizing stopping times may be taken as T_n, but if $\mathscr{E} N_i(t)<\infty$ for all i and t also as constants.

2. Random Right Censorship of a Markov Renewal Process

We suppose that it is not possible to observe complete realizations of J_0, X_1, $J_1,\ldots$ but rather that observations consist of the values taken by the processes Z_-, L and $\Delta \tilde{N}_{ij}\,(i,j\leqq m)$ on some random subset of the time axis. Letting K be the indicator process of this set, then we say that the (Markov renewal) process is under observation at time t if $K(t)=1$, when the state at $t-$, how long it has been occupied, and a possible transition at that time instant and the new state can be observed; otherwise $K(t)=0$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space on which the process K and the r.v.'s J_0, X_1, $J_1,\ldots$ are defined. Then the following assumptions about K are made, and with the exception of A 3* are taken to be in force throughout the rest of the paper, unless explicitly stated otherwise:

A 1. There exist r.v.'s T_n such that almost surely,

$$S_n \leqq T_n \leqq S_{n+1}\ \forall n \quad \text{and} \quad K(t)=\sum_n \chi_{(S_n, T_n]}(t)\ \forall t.$$

A 2. There exists a sub σ-algebra $\mathscr{A}$ of $\mathscr{F}$ containing all P-null sets of $\mathscr{F}$, conditionally independent of $\sigma\{\tilde{N}_{ij}(s);\, i,j\leqq m,\, s\in[0,\infty)\}$ given J_0, and such that for each n T_n (see A 1) is a stopping time with respect to the family of σ-algebras $\{\mathscr{F}_t;\, t\in[0,\infty)\}$ defined by

$$\mathscr{F}_t = \mathscr{A} \vee \sigma\{J_0, \tilde{N}_{ij}(s);\, i,j\leqq m,\, s\in[0,t]\}.$$

A 3. $\mathscr{E}(\#\{n: T_n>S_n\})<\infty$.

A 3*. $\mathscr{E}((\#\{n: T_n>S_n\})^{7+\varepsilon})<\infty$ for some $\varepsilon>0$.

The first assumption limits us to what might be called "right-censorship". A 2 implies that the process K is *predictable* and might be interpreted as stating that $K(t)$ does not depend on the development of the Markov renewal process in $[t,\infty)$ given the past at time t. According to A 3 or A 3* the number of at least partially observed sojourn times is certainly almost surely finite. A 3 is sufficient for proving consistency of our estimators, and weak convergence when the X_i's take on values in $\mathbb{N}$; for general Q_{ij} we need A 3* in proving weak convergence. It ought to be possible to weaken A 3* by using perhaps a different method of proof: however in practical applications A 3 and A 3* will generally hold. For suppose that for some fixed finite k and t observation always stops after at most k transitions after the time instant t; i.e. $s>S_{\tilde{N}(t)+k}$

$\Rightarrow K(s)=0$. In this case $\#\{n: T_n > S_n\} \leqq \tilde{N}(t)+k$ which has finite moments of all orders.

We mention briefly some special cases. If $K(t)=\chi_{[0,T]}(t)$ for some r.v. T which is conditionally independent of $X_1, J_1, \ldots$ given J_0, then A1 and A2 hold. If $m=1$, and $K(t)=\chi_{[0,U_r]}(L(t))$ for $t\in(S_{r-1}, S_r]$, $r=1,\ldots,n$ and zero on (S_n, ∞) for a fixed $n\geqq 1$ and positive r.v.'s $U_1,\ldots,U_n$ independent of $X_1, X_2, \ldots$ then all the assumptions hold and we have in fact the usual model of n censored observations of $X_1,\ldots,X_n$ with censoring variables $U_1,\ldots,U_n$.

Next we introduce processes which count observed sojourn times: for $u\geqq 0$ we define

$$\begin{aligned} N_{ij}(u) &= \#\{n\geqq 1: J_{n-1}=i, J_n=j, X_n\leqq u, K(S_n)=1\} \\ &= \text{number of sojourn times in state } i \text{ observed to take on a} \\ &\quad \text{value } \leqq u \text{ and to be followed by a jump to state } j \\ &= \#\{t: \Delta\tilde{N}_{ij}(t)=1,\ Z(t-)=i,\ L(t)\leqq u,\ K(t)=1\}, \\ N_i(u) &= \sum_j N_{ij}(u). \end{aligned} \tag{12}$$

For $u>0$ we define

$$\begin{aligned} Y_i(u) &= \#\{n\geqq 1: J_{n-1}=i, X_n\geqq u, K(S_{n-1}+u)=1\} \\ &= \text{number of sojourn times in state } i \text{ observed to take on a value } \geqq u \\ &= \#\{t: Z(t-)=i, L(t)=u, K(t)=1\}, \\ Y_i(0) &= Y_i(0+), \\ Y(u) &= \sum_i Y_i(u) \quad \text{(also for } u=0). \end{aligned} \tag{13}$$

Note that $Y(0)=\#\{n: T_n > S_n\}$.

By A1 and A3, $\mathscr{E} Y_i(u)$ and $\mathscr{E} N_{ij}(u) \leqq \mathscr{E} Y(0) < \infty$ for all i, j and u. The sample paths of the processes N_{ij} are almost surely zero at time zero, right continuous with left hand limits, nondecreasing and integer valued; while the sample paths of Y_i are nonincreasing, left continuous with right hand limits, and nonnegative integer valued. According to the interpretation of K, these processes are observable.

By the counting process results of Sect. 1 applied to $\{\tilde{N}_{ij}; i,j\leqq m\}$, if we define

$$\begin{aligned} A_{ij}(t) &= \int_0^t \chi_{\{i\}}(Z(s-)) \frac{dQ_{ij}(L(s))}{1-H_i(L(s)-)} \\ &= \sum_{r=1}^{\tilde{N}(t-)} \chi_{\{i\}}(J_{r-1}) \int_0^{X_r} \frac{dQ_{ij}(u)}{1-H_i(u-)} + \chi_{\{i\}}(J_{\tilde{N}(t-)}) \int_0^{L(t)} \frac{dQ_{ij}(u)}{1-H_i(u-)} \end{aligned} \tag{14}$$

and

$$M_{ij}=\tilde{N}_{ij}-A_{ij}, \tag{15}$$

it follows that with respect to $\{\mathscr{F}_t\}$ (defined in A2) the M_{ij}'s are local square integrable martingales, zero at time zero, with

$$\begin{aligned}
&\langle M_{ij}, M_{ij}\rangle = \int (1-\Delta A_{ij})\, d A_{ij} \\
&\langle M_{ij}, M_{ij'}\rangle = -\int \Delta A_{ij}\, d A_{ij'}, \qquad j \neq j' \\
&\langle M_{ij}, M_{i'j'}\rangle = 0 \qquad\qquad i \neq i'.
\end{aligned} \tag{16}$$

Here the localizing stopping times may be taken as constants by finiteness of $\mathscr{E}\tilde{N}_{ij}(t)$ for all i, j and t.

Now by A 3, it follows that for all i and j

$$\mathscr{E}\int_0^\infty K(s)\, d\tilde{N}_{ij}(s) \leqq \mathscr{E}\, Y(0) < \infty. \tag{17}$$

But because K is predictable, nonnegative and bounded, by the stochastic integral results of Sect. 1,

$$\mathscr{E}\int_0^t K(s)\, d\tilde{N}_{ij}(s) = \mathscr{E}\int_0^t K(s)\, d A_{ij}(s) \quad \text{for all } t < \infty. \tag{18}$$

So letting $t \to \infty$,

$$\mathscr{E}\int_0^\infty K(s)\, d A_{ij}(s) < \infty. \tag{19}$$

Next, since

$$\begin{aligned}
\mathscr{E}\left(\int_0^t K(s)\, d M_{ij}(s)\right)^2 &= \mathscr{E}\int_0^t K(s)^2\, d\langle M_{ij}, M_{ij}\rangle(s) \\
&= \mathscr{E}\int_0^t K(s)(1-\Delta A_{ij}(s))\, d A_{ij}(s),
\end{aligned} \tag{20}$$

by letting $t \uparrow \infty$, we find

$$\lim_{t\uparrow\infty} \mathscr{E}\left(\int_0^t K(s)\, d M_{ij}(s)\right)^2 < \infty. \tag{21}$$

But since $\int K\, d M_{ij}$ is a martingale w.r.t. $\{\mathscr{F}_t\}$ this shows that $\int K\, d M_{ij}$ is in fact a (zero mean) square integrable martingale, clearly with

$$\langle \int K\, d M_{ij}, \int K\, d M_{i'j'}\rangle = \int K\, d\langle M_{ij}, M_{i'j'}\rangle. \tag{22}$$

Next, as in Gill [6, Lemma 3 and Proposition 4] (where similar results are obtained in a rather special case), we find that for any bounded measurable functions f and f' on $[0, \infty)$

$$\int_0^\infty f(u)\, d N_{ij}(u) = \int_0^\infty f(L(t))\, d\tilde{N}_{ij}(t), \tag{23}$$

$$\int_0^\infty f(u)\, Y_i(u) \frac{d Q_{ij}(u)}{1-H_i(u-)} = \int_0^\infty f(L(t))\, K(t)\, d A_{ij}(t), \tag{24}$$

and so defining

$$\begin{aligned}
Z_{ij}(v) &= N_{ij}(v) - \int_0^v Y_i(u) \frac{d Q_{ij}(u)}{1-H_i(u-)} \\
Z_i(v) &= \sum_j Z_{ij}(v),
\end{aligned} \tag{25}$$

we see that

$$\int_0^\infty f(L(s))\,K(s)\,dM_{ij}(s)=\int_0^\infty f(u)\,dZ_{ij}(u), \tag{26}$$

where $f(L)$ is a bounded predictable process (L is left continuous with right hand limits and adapted). So by the theory of stochastic integrals, taking expectations in (26),

$$0=\mathscr{E}\int_0^\infty f(u)\,dZ_{ij}(u)=\int_0^\infty f(u)\,d\mathscr{E}N_{ij}(u)-\int_0^\infty f(u)\,\mathscr{E}Y_i(u)\frac{dQ_{ij}(u)}{1-H_i(u-)} \tag{27}$$

and similarly (by (9) and (26))

$$\begin{aligned}
&\mathscr{E}\left(\left(\int_0^\infty f(u)\,dZ_{ij}(u)\right)\left(\int_0^\infty f'(u)\,dZ_{i'j'}(u)\right)\right)\\
&\quad=\mathscr{E}\int_0^\infty f(L(s))\,f'(L(s))\,K(s)\,d\langle M_{ij},M_{i'j'}(s)\rangle\\
&\quad=\begin{cases}
\displaystyle\int_0^\infty f(u)\,f'(u)\,\mathscr{E}Y_i(u)\left(1-\frac{\Delta Q_{ij}(u)}{1-H_i(u-)}\right)\frac{dQ_{ij}(u)}{1-H_i(u-)}, & i=i',\ j=j'\\
\displaystyle-\int_0^\infty f(u)\,f'(u)\,\mathscr{E}Y_i(u)\frac{\Delta Q_{ij}(u)}{1-H_i(u-)}\frac{dQ_{ij'}(u)}{1-H_i(u-)}, & i=i',\ j\neq j'\\
0, & i\neq i'
\end{cases}
\end{aligned} \tag{28}$$

(where the last step follows by similar equalities to (24)).

3. Estimation of Transition Probabilities Q_{ij}

Suppose we are given n independent identically distributed observations of N_{ij} and Y_i. Let N_{ij}^n, Y_i^n, N_i^n, Z_{ij}^n, Z_i^n denote the sums of the n realizations of N_{ij}, Y_i, ... (see definitions (12), (13) and (25)). We build up estimators of Q_{ij} in two steps, first estimating H_i by $\hat{H}_i^n$, defined by

$$\hat{H}_i^n(t)=1-\prod_{s\leq t}\left(1-\frac{\Delta N_i^n(s)}{Y_i^n(s)}\right), \tag{29}$$

where by convention (which we adopt from now on) $0/0=0$. An equivalent implicit definition is

$$\hat{H}_i^n(t)=\int_0^t(1-\hat{H}_i^n(s-))\frac{dN_i^n(s)}{Y_i^n(s)}. \tag{30}$$

Now we can define $\hat{Q}_{ij}^n$ by

$$\hat{Q}_{ij}^n(t)=\int_0^t(1-\hat{H}_i^n(s-))\frac{dN_{ij}^n(s)}{Y_i^n(s)}. \tag{31}$$

Adding over j we see that $\hat{H}_i^n = \sum_j \hat{Q}_{ij}^n$ (cf. (4)). These definitions can be motivated by the following facts. Putting in (27) $f = \chi_{[0,t]}$ we find

$$\mathscr{E} N_{ij}(t) = \int_0^t \mathscr{E} Y_i(s)(1 - H_i(s-))^{-1} \, dQ_{ij}(s)$$

which implies, adding over j, that

$$\mathscr{E} N_i(t) = \int_0^t \mathscr{E} Y_i(s)(1 - H_i(s-))^{-1} \, dH_i(s).$$

So if t satisfies $\mathscr{E} Y_i(t) > 0$, which implies that $\mathscr{E} Y_i(s) > 0$ and $1 - H_i(s-) > 0$ on $[0, t]$, we find

$$H_i(t) = \int_0^t (1 - H_i(s-)) \frac{d\mathscr{E} N_i(s)}{\mathscr{E} Y_i(s)} \tag{32}$$

(cf. (30)). We shall see in Lemma 1 that such an expression defines H_i uniquely. Similarly, also supposing $\mathscr{E} Y_i(t) > 0$, we find

$$Q_{ij}(t) = \int_0^t (1 - H_i(s-)) \frac{d\mathscr{E} N_{ij}(s)}{\mathscr{E} Y_i(s)} \tag{33}$$

(cf. (31)). Of course we can't expect to be able to estimate Q_{ij} outside of $\{t: \mathscr{E} Y_i(t) > 0\}$.

As a referee remarked, one might also be interested in the analogue of the well known "Nelson plot" (see e.g. Aalen [1, Sect. 6.1]), namely $\int_0^t Y_i^n(s)^{-1} \, dN_{ij}^n(s)$, which estimates the function $\int_0^t (1 - H_i(s-))^{-1} \, dQ_{ij}(s)$. Its properties can be derived in exactly the same way as those of $\hat{Q}_{ij}$ and $\hat{H}_i$ (in fact many of the arguments become easier). However we shall not go into the details here.

We first prove consistency, for which we need two lemmas.

Lemma 1. *Let A and B be right continuous nondecreasing functions on $[0, \infty)$, zero at time zero; suppose $\Delta A \leqq 1$ and $\Delta B < 1$ on $[0, \infty)$. Then the unique solution Z of*

$$Z(t) = 1 - \int_0^t \frac{Z(s-)}{1 - \Delta B(s)} (dA(s) - dB(s)) \tag{34}$$

which is locally bounded (i.e. bounded on $[0, t]$ for each $t < \infty$) and right continuous with left hand limits is given by

$$Z(t) = \frac{\prod_{s \leqq t} (1 - \Delta A(s)) \cdot \exp(-A_c(t))}{\prod_{s \leqq t} (1 - \Delta B(s)) \cdot \exp(-B_c(t))},$$

where $A_c(t) = A(t) - \sum_{s \leqq t} \Delta A(s)$, etc.

Proof. The proof given by Liptser and Shiryaev [10, Lemma 18.8] for the case $B=0$ goes through exactly, replacing their a with $(1-\Delta B)^{-1}$, A with $A-B$, $Z(0)$ with 1, and α with $\int(1-\Delta B)^{-1}(dA+dB)$. □

Corollary. *For all t such that $1-H_i(t)>0$*

$$1-H_i(t)=\prod_{s\leqq t}\left(1-\frac{\Delta H_i(s)}{1-H_i(s-)}\right)\exp\left(-\int_0^t\frac{dH_{ic}(s)}{1-H_i(s-)}\right) \tag{36}$$

and

$$\frac{1-\hat{H}_i^n(t)}{1-H_i(t)}=1-\int_0^t\frac{1-\hat{H}_i^n(s-)}{1-H_i(s-)}\left(1-\frac{\Delta H_i(s)}{1-H_i(s-)}\right)^{-1}\left(\frac{dN_i^n(s)}{Y_i^n(s)}-\frac{dH_i(s)}{1-H_i(s-)}\right). \tag{37}$$

Proof. Since (34) holds with $B\equiv 0$, $A=\int(1-H_{i-})^{-1}dH_i$ and $Z=1-H_i$ we can substitute these quantities in (35) which gives (36). Thus by (22) and (29), (35) holds with $A=\int(Y_i^n)^{-1}dN_i^n$, $B=\int(1-H_{i-})^{-1}dH_i$, and $Z=(1-\hat{H}_i^n)/(1-H_i)$; substituting them in (34) gives (37). □

Note that if $Y_i^n(t)>0$, if $1-H_i(t-)>0$ and if $1-H_i(t)=0$ then almost surely $\hat{H}_i^n(t)=H_i(t)=1$; also note that $1-\Delta H_i/(1-H_{i-})=(1-H_i)/(1-H_{i-})$. So we can rewrite (37) as

$$\hat{H}_i^n-H_i=(1-H_i)\int\frac{1-\hat{H}_{i-}^n}{1-H_i}\frac{n}{Y_i^n}\frac{dZ_i^n}{n} \tag{38}$$

on $\{t\colon Y_i^n(t)>0$ and $1-H_i(t-)>0\}$, where Z_i was defined in (25) and the convention $0/0=0$ may have to be invoked. Next we write, also on $\{t\colon Y_i^n(t)>0$ and $1-H_i(t-)>0\}$

$$\begin{aligned}\hat{Q}_{ij}^n-Q_{ij}&=\int(1-\hat{H}_{i-}^n)\left(\frac{dN_{ij}^n}{Y_i^n}-\frac{dQ_{ij}}{1-H_{i-}}\right)+\int\left(\frac{1-\hat{H}_{i-}^n}{1-H_{i-}}-1\right)dQ_{ij}\\&=\int(1-\hat{H}_{i-}^n)\frac{n}{Y_i^n}\frac{dZ_{ij}^n}{n}-\int\left(\int^-\frac{1-\hat{H}_{i-}^n}{1-H_i}\frac{n}{Y_i^n}\frac{dZ_i^n}{n}\right)dQ_{ij}\\&=\int(1-\hat{H}_{i-}^n)\frac{n}{Y_i^n}\frac{dZ_{ij}^n}{n}-Q_{ij}\int^-\frac{1-\hat{H}_{i-}^n}{1-H_i}\frac{n}{Y_i^n}\frac{dZ_i^n}{n}\\&\quad+\int^-Q_{ij}\frac{1-\hat{H}_{i-}^n}{1-H_i}\frac{n}{Y_i^n}\frac{dZ_i^n}{n},\end{aligned} \tag{39}$$

where we have used (31), (37) and integration by parts. (Adding over j and integrating by parts gives (38) again of course!) □

The next lemma and its corollary give conditions under which expressions such as the right hand sides of (38) and (39) converge uniformly in t to zero, in probability: this will be the consistency result of Theorem 1.

Lemma 2. *Suppose H_n and Z_n are processes on $[0,\tau]$ whose sample paths are almost surely right continuous with left hand limits, of bounded variation, and satisfying $Z_n(0)=0$; $n=1,2,\ldots$. Suppose $\|Z_n\|=\sup_{t\in[0,\tau]}|Z_n(t)|\to_P 0$. If*

$\int_{(0,\tau)} |dH_n(t)|$ *is bounded in probability as* $n\to\infty$, *then* $\|\int H_{n-}\,dZ_n\|\to_P 0$ *and* $\|\int^- H_n\,dZ_n\|\to_P 0$; *if* $\int_{(0,\tau]} |dH_n(t)|$ *is bounded in probability, then* $\|\int H_n\,dZ_n\|\to_P 0$.

Proof. We can write

$$(\int H_{n-}\,dZ_n)(t)=\int_{(0,t)}(Z_n(t)-Z_n(s))\,dH_n(s)$$

and

$$(\int H_n\,dZ_n)(t)=\int_{(0,t]}(Z_n(t)-Z_n(s-))\,dH_n(s).$$

So $\|\int H_{n-}\,dZ_n\|$ and $\|\int^- H_n\,dZ_n\|\leqq 2\|Z_n\|\int_{(0,\tau)}|dH_n(t)|\to_P 0$ as $n\to\infty$ and similarly $\|\int H_n\,dZ_n\|\to_P 0$. □

Corollary. *Suppose Lemma 2 allows us to conclude that* $\|\int H_n^{(i)}\,dZ_n\|\to_P 0$, $i=1,\ldots,r$, *and* $\|\int H_{n-}^{(i)}\,dZ_n\|\to_P 0$, $i=r+1,\ldots,s$. *Then*

$$\left\|\int\prod_{i=1}^{r}H_n^{(i)}\cdot\prod_{i=r+1}^{s}H_{n-}^{(i)}\,dZ_n\right\|\to_P 0,$$

as $n\to\infty$.

Proof. Apply Lemma 2 first to $H_n^{(1)}$ and Z_n, then to $H_n^{(2)}$ and $\int H_n^{(1)}\,dZ_n$, etc. □

Theorem 1. *Let* $\tau_i=\sup\{t:\ \mathscr{E}\,Y_i(t)>0\}$. *Then as* $n\to\infty$

$$\sup_{t\in[0,\tau_i]}|\hat{H}_i^n(t)-H_i(t)|\to_P 0$$

and

$$\sup_{t\in[0,\tau_i]}|\hat{Q}_{ij}^n(t)-Q_{ij}(t)|\to_P 0,$$

unless $\mathscr{E}\,Y_i(\tau_i)=0$ *and* $\Delta H_i(\tau_i)$ *or* $\Delta Q_{ij}(\tau_i)>0$, *in which case* $[0,\tau_i]$ *must be replaced by* $[0,\tau_i)$ *in the corresponding supremum.*

Proof. By the weak law of large numbers and monotonicity arguments it is easy to show that $\|n^{-1}N_{ij}^n-\mathscr{E}\,N_{ij}\|_\infty\to_P 0$ and

$$\|\int n^{-1}Y_i^n(1-H_{i-})^{-1}\,dQ_{ij}-\int\mathscr{E}\,Y_i(1-H_{i-})^{-1}\,dQ_{ij}\|_\infty\to_P 0.$$

So $\|n^{-1}Z_{ij}^n\|_\infty\to_P 0$ (using (27) with $f=\chi_{[0,v]}$). Suppose $\tau\leqq\tau_i$ is such that $\mathscr{E}(Y_i(\tau))>0$. Then $\int_{(0,\tau)}|d(n/Y_{i+}^n(t)|\leqq n/Y_i^n(\tau)$, bounded in probability as $n\to\infty$, and so by the corollary to Lemma 2 and (39) $\|\hat{Q}_{ij}^n-Q_{ij}\|_\tau\to_P 0$ as $n\to\infty$; adding over j shows $\|\hat{H}_i^n-H_i\|_\tau\to_P 0$. If $\mathscr{E}\,Y_i(\tau_i)>0$ we are ready. Otherwise monotonicity arguments show that the required results hold with $[0,\tau_i)$ in place of $[0,\tau_i]$; if $\Delta H_i(\tau_i)=0$ or $\Delta Q_{ij}(\tau_i)=0$ then adding τ_i to the range of the supremum in the corresponding term changes nothing. □

Actually almost sure convergence in the appropriate set up is also easy to derive using the strong law of large numbers and Glivenko-Cantelli type arguments, and a suitable modification of Lemma 2.

Now we turn to proving weak convergence, giving first a result for the case that Q_{ij}'s give weight only to the positive integers. Again we use the representations (38) and (39), and also need Theorem 1.

Theorem 2. *Suppose $X_1, X_2, \ldots$ take values in $\mathbb{N}$. Let $\tau_i = \sup\{t \in \mathbb{N}: \mathscr{E} Y_i(t) > 0\}$ ($\tau_i \in \mathbb{N} \cup \{\infty\}$). Then $\{n^{\frac{1}{2}}(\hat{Q}_{ij}^n(t) - Q_{ij}(t)), n^{\frac{1}{2}}(\hat{H}_i^n(t) - H_i(t)); t \in \mathbb{N}, t \leqq \tau_i, i, j \leqq m\}$ is distributed asymptotically as*

$$\left\{ \sum_{s=1}^{t} \frac{1-H_i(s-)}{\mathscr{E} Y_i(s)} U_{ij}(s) - \sum_{s=1}^{t-1} \frac{(Q_{ij}(t)-Q_{ij}(s))}{\mathscr{E} Y_i(s)} \frac{1-H_i(s-)}{1-H_i(s)} U_i(s), \right.$$

$$\left. (1-H_i(t)) \sum_{s=1}^{t} \frac{1-H_i(s-)}{1-H_i(s)} \frac{1}{\mathscr{E} Y_i(s)} U_i(s);\ t \in \mathbb{N},\ t \leqq \tau_i,\ i, j \leqq m \right\},$$

where the $U_{ij}(s)$'s are multivariate normally distributed r.v.'s with expectations zero; $U_i(s) = \sum_j U_{ij}(s)$; and

$$\operatorname{var}(U_{ij}(s)) = \mathscr{E} Y_i(s) \left(1 - \frac{\Delta Q_{ij}(s)}{1-H_i(s-)}\right) \frac{\Delta Q_{ij}(s)}{1-H_i(s-)},$$

$$\operatorname{cov}(U_{ij}(s), U_{ij'}(s)) = -\mathscr{E} Y_i(s) \frac{\Delta Q_{ij}(s)}{1-H_i(s-)} \frac{\Delta Q_{ij'}(s)}{1-H_i(s-)} \qquad j \neq j',$$

$$\operatorname{cov}(U_{ij}(s), U_{i'j'}(s')) = 0 \qquad i \neq i' \text{ or } s \neq s'.$$

Proof. Multiply (38) and (39) by $n^{\frac{1}{2}}$ and rewrite the integrals in the right hand sides as sums over $s \leqq t$ or $s < t$ as appropriate, with $n^{-\frac{1}{2}} dZ_{ij}^n(s)$ replaced with $n^{-\frac{1}{2}} \Delta Z_{ij}^n(s)$. By Theorem 1 and convergence in probability of $n^{-1} Y_i^n(s)$ the coefficients of $n^{-\frac{1}{2}} \Delta Z_{ij}^n(s)$ all converge in probability, while by the central limit theorem and (27) and (28) with $f = \chi_{\{s\}}$, $f' = \chi_{\{s'\}}$,

$$\{n^{-\frac{1}{2}} \Delta Z_{ij}^n(s);\ i, j \leqq m,\ s \in \mathbb{N}\} \to_{\mathscr{D}} \{U_{ij}(s);\ i, j \leqq m,\ s \in \mathbb{N}\}$$

and the theorem is proved. □

Apart from the fact that the final theorem on weak convergence with arbitrary Q_{ij} uses A3*, it includes Theorem 2 as a special case. The method of proof is essentially the same, though many more technical details are encountered.

Theorem 3. *Suppose* A3* *holds, and choose τ_i, $i \leqq m$, such that $\mathscr{E} Y_i(\tau_i) > 0$. Then considered as a random element of $\prod_i (D[0, \tau_i])^{m+1}$ (see Billingsley [3]) $\{n^{\frac{1}{2}}(\hat{Q}_{ij} - Q_{ij}), n^{\frac{1}{2}}(\hat{H}_i - H_i);\ i, j \leqq m\}$ is asymptotically distributed as*

$$\left\{ \int \frac{1-H_{i-}}{\mathscr{E} Y_i} dW_{ij} - Q_{ij} \int^{-} \frac{1-H_{i-}}{1-H_i} \frac{1}{\mathscr{E} Y_i} dW_i + \int^{-} Q_{ij} \frac{1-H_{i-}}{1-H_i} \frac{1}{\mathscr{E} Y_i} dW_i, \right.$$

$$\left. (1-H_i) \int \frac{1-H_{i-}}{1-H_i} \frac{1}{\mathscr{E} Y_i} dW_i;\ i, j \leqq m \right\},$$

where the W_{ij}'s are jointly zero mean Gaussian processes with independent multivariate increments, the sets $\{W_{ij};\ j \leqq m\}$, $i=1,\ldots,m$, being independent of one another; $W_i = \sum_j W_{ij}$; and

$$\operatorname{var}(W_{ij}(t)) = \int_0^t \mathscr{E}\, Y_i(s) \left(1 - \frac{\Delta Q_{ij}(s)}{1-H_i(s-)}\right) \frac{dQ_{ij}(s)}{1-H_i(s-)},$$

$$\operatorname{cov}(W_{ij}(t), W_{ij'}(t)) = -\int_0^t \mathscr{E}\, Y_i(s) \frac{\Delta Q_{ij}(s)}{1-H_i(s-)} \frac{dQ_{ij'(s)}}{1-H_i(s-)}.$$

The integrals with respect to W_{ij} and W_i are stochastic integrals in the sense of Meyer [11] *(the W_{ij}'s are square integrable martingales with respect to the natural family of σ-algebras) or can equivalently be defined by formal integration by parts, the resulting expressions having a pathwise definition.*

Proof. We only sketch the proof here; details are given as three lemmas in Sect. 5. The proof is again based on the representations (38) and (39). Multiplying these equations by $n^{\frac{1}{2}}$, we note that the expressions on the right hand sides consist of integrals, where the integrands are products of fixed functions and the processes $1-\hat{H}_{i-}^n$ and n/Y_i^n, which, in probability, converge uniformly on $[0, \tau_i]$ to $1-H_{i-}$ and $(\mathscr{E}\, Y_i)^{-1}$. The integrals are taken with respect to $n^{-\frac{1}{2}} Z_{ij}^n$ and $n^{-\frac{1}{2}} Z_i^n = \sum_j n^{-\frac{1}{2}} Z_{ij}^n$. Now these processes have finite dimensional distributions which converge to those of W_{ij} and W_i by using the central limit theorem and (27) and (28) with $f = \chi_{[0,t]}$ and $f' = \chi_{[0,t']}$. In Lemma 3 in Sect. 5 we prove tightness of $n^{-\frac{1}{2}} Z_{ij}^n$ in $D[0, \tau_i]$, here A 3* is used. Actually we need to prove a little more, because in the next step of the proof we want to apply the Skorohod-Dudley theorem (see e.g. Pyke [13]) and consider processes $(\hat{H}_i^n - H_i)'$, $(n^{-1} Y_i^n - \mathscr{E}\, Y_i)'$, and $(n^{-\frac{1}{2}} Z_{ij}^n)'$ $(i, j \leqq m,\ n=1,2,\ldots)$ defined on a new probability space with the same joint distribution for each n as their unprimed equivalents, and converging almost surely in the supremum norm to 0, 0, and W_{ij}' respectively (the W_{ij}''s also having the same joint distribution as the W_{ij}'s). The construction is possible with the supremum norm distance rather than the Skorohod distance, if the sample paths of each W_{ij} can be taken to be continuous with probability one; i.e. if the Q_{ij}'s are continuous. However, we can get round this problem by inserting a time interval at each jump point t of Q_{ij}, joining up $n^{-\frac{1}{2}} Z_{ij}^n(t-)$ to $n^{-\frac{1}{2}} Z_{ij}^n(t)$ with a straight line across this interval, and proving joint weak convergence of these new processes on the resulting extended time interval to the corresponding objects obtained from the W_{ij}'s, which can be taken to be continuous. Of course we shall need weak convergence of the finite dimensional distributions of $n^{-\frac{1}{2}} Z_{ij-}^n$ and $n^{-\frac{1}{2}} Z_{ij}^n$; which again follows from the central limit theorem and (27) and (28). In Lemma 4 we show that the above programme can indeed be carried out. Now we are at liberty to apply the Skorohod-Dudley theorem (after which we remove the extra intervals again). Finally the corollary to Lemma 5 shows that the suprema over $[0, \tau]$ of the absolute difference between the primed versions of $n^{\frac{1}{2}}$ times (38) and (39) and their "obvious" limits (given in the statement of

the theorem above) converge almost surely to zero, where some care is needed because these obvious limits cannot be defined directly as pathwise integrals if the Q_{ij}'s have continuous components. This problem is also resolved in Lemma 5. □

4. Remarks and Generalizations

Though the limiting covariances of $n^{\frac{1}{2}}(\hat{Q}_{ij}^n - Q_{ij})$ can be consistently estimated, it seems difficult to use them to construct confidence bands. At least this is not the case for $n^{\frac{1}{2}}(\hat{H}_i^n - H_i)$; e.g. Gill [6 or 7]. An advantage of the Nelson plot mentioned earlier (immediately before Lemma 1) is that confidence bands for them are easy to construct, because the corresponding asymptotic process will have independent increments; i.e. we will obtain a deterministically time transformed Brownian motion. Also Theorem 3 does not give us in general a limiting distribution of the estimators of the transition probabilities $Q_{ij}(\infty)$ of the Markov chain $J_0, J_1, \ldots$ associated with the Markov renewal process, supposing that estimation should be possible (Theorem 1 gives conditions for consistent estimation).

Theorem 3 specialized to the case $m=1$ and assuming a rather special form for the censoring process K gives us a weak convergence theorem for the so-called product limit estimator. This result generalizes that obtained by Breslow and Crowley [4], for which continuous distributions of both censoring variables and the variables of interest were assumed. In Gill [7] we derive this result in a more direct fashion and in more generality.

We next discuss possible generalizations of our results. Firstly, can we drop the restriction A1 to right censorship? In the discrete case this gives no problems: we must assume that $\mathscr{E} Y_i(s) < \infty$ for all i and $s = 1, 2, \ldots$ and realize that we can only estimate $\Delta Q_{ij}(s)/(1 - H_i(s-))$ for s such that $\mathscr{E} Y_i(s) > 0$ (where Y_i is still defined by (13)). However in the general case we have made strong use of many of the properties of the processes Y_i as the following list of "corrections" shows: replace A1 with the *assumption* that the sample paths of Y_i are left continuous with right hand limits; in A2 instead of the stopping time condition assume that K is predictable with respect to the given σ-algebras; and in A3 and A3* replace $\#\{n: T_n > S_n\}$ with $\#\{n: K(s) = 1$ for some $s \in (S_n, S_{n+1}]\}$. Theorem 1 then remains true if we replace τ_i with $\tau_i = \sup\{t: \mathscr{E} \int_{(0,t)} |dY_{i+}(s)| < \infty$ and $\mathscr{E} \min_{s \in (0,t]} Y_i(s) > 0\}$ and modify the condition "unless $\mathscr{E} Y_i(\tau_i) = 0$" accordingly. Theorem 3 remains valid if we choose τ_i such that $\mathscr{E} \int_{(0,\tau_i)} |dY_{i+}(t)| < \infty$, $\mathscr{E} \min_{s \in (0,\tau_i]} Y_i(s) > 0$, $\|n^{-1} Y_i^n - \mathscr{E} Y_i\|_{\tau_i} \to_p 0$ and $\int_{(0,\tau_i)} |d\mathscr{E} Y_{i+}(t)| < \infty$, where actually the last two properties are a consequence of the first one.

Alternatively, what happens when we drop the assumption of identically distributed observations of $\{N_{ij}, Y_i\}$? Consider a triangular array $\{N_{ij}^{kn}, Y_i^{kn}; i, j \leqq m\}$, $k = 1, \ldots, n$; $n = 1, 2, \ldots$, for each (k, n) defined as in Sects. 1 and 2 with fixed Q_{ij}'s, but possibly differing initial distributions and "censoring distri-

butions", and independent over k for each n. Define

$$N_{ij}^n = \sum_{k=1}^{n} N_{ij}^{kn}$$

and

$$Y_i^n = \sum_{k=1}^{n} Y_i^{kn}$$

and proceed as in Sect. 3. It is not too difficult to see that if we replace A3 with the assumption that for some fixed C, $\mathcal{E} Y^{kn}(0) < C < \infty$ for all k and n and similarly modify A3*, and *define* $\mathcal{E} Y_i$ to be $\lim_{n\to\infty} n^{-1} \mathcal{E} Y_i^n$, which we assume to exist and to be left continuous with right hand limits, then Theorems 1 to 3 still hold. Important for applications to medical trials is the fact that even if for some states i the transition probabilities vary with n and k, for the other states all our results go through.

Finally, instead of looking at n independent observations, suppose that a single Markov renewal process is, with censoring, observed over an expanding sequence of time intervals. As a specific example, consider the model of Sects. 1 and 2 where we drop assumptions A1 and A3 but do suppose that the X_i's take values in $\mathbb{N}$. In A2 we assume that K is predictable instead of making the stopping times assumption. Defining

$$Y_i^n(u) = \#\{t \leqq n\colon Z(t-) = i,\ L(t) = u,\ K(t) = 1\}$$

and

$$\Delta N_{ij}^n(u) = \#\{t \leqq n\colon \Delta \tilde{N}_{ij}(t) = 1,\ Z(t-) = i,\ L(t) = u,\ K(t) = 1\}$$

(cf. (12) and (13)) it is possible, as in Bather [2, Lemma 1], to apply Chow [5, Theorem 5] to show that on the set of ω for which $Y_i^n(u) \to \infty$ as $n \to \infty$ (for fixed i and u), $\Delta N_{ij}^n(u)/Y_i^n(u) \to \Delta Q_{ij}(u)/(1 - H_i(u-))$ a.s. It is not yet clear to the author what can be done for general Q_{ij}'s, nor indeed in this special case how weak convergence can be proved.

It should be pointed out that we could not apply the general theory of Aalen [1] to derive our results, despite the strong similarity of models. This is because of the occurrence of the process L in formula (14), whose saw-tooth paths prevent the transformation (26) from M_{ij} to Z_{ij} (defined in (15) and (25)) from preserving the martingale property of M_{ij}. However, it preserves enough of it (the properties of first and second moments) for our asymptotic results.

5. Technical Lemmas Needed for Proof of Theorem 3

Lemma 3. *For each i, j and τ such that $\mathcal{E} Y_i(\tau) > 0$, $n^{-\frac{1}{2}} Z_{ij}^n$ is tight in $D[0, \tau]$ as $n \to \infty$ if* A3* *holds.*

Proof. Let I_1 and I_2 be the intervals $(t_1, t]$ and $(t, t_2]$ for some time instants $0 \leqq t_1 < t < t_2 \leqq \tau$. Write $\Delta_k X$ for $\int_{I_k} dX$, $k = 1$ or 2, for a process or function X of

bounded variation. We show that we can find $C>0$ and $\alpha\in(\frac{1}{2}, 1)$ such that

$$n^{-2}\mathscr{E}(\Delta_1 Z_{ij}^n\cdot\Delta_2 Z_{ij}^n)^2\leqq C(\Delta_1 Q_{ij}\cdot\Delta_2 Q_{ij})^\alpha, \tag{40}$$

which proves tightness in view of Billingsley [3, Theorem 15.6 and the remarks on p. 133]. Now $(\Delta_1 Z_{ij}^n, \Delta_2 Z_{ij}^n)$ is a sum of n independent random variables with zero means (by (27) with $f=\chi_{I_k}$) each distributed as $(\Delta_1 Z_{ij}, \Delta_2 Z_{ij})$. So

$$\begin{aligned}n^{-2}\mathscr{E}(\Delta_1 Z_{ij}^n\cdot\Delta_2 Z_{ij}^n)^2 &= n^{-1}\mathscr{E}(\Delta_1 Z_{ij}\cdot\Delta_2 Z_{ij})^2\\ &\quad+((n-1)/n)\mathscr{E}(\Delta_1 Z_{ij})^2\,\mathscr{E}(\Delta_2 Z_{ij})^2\\ &\quad+2((n-1)/n)(\mathscr{E}(\Delta_1 Z_{ij}\cdot\Delta_2 Z_{ij}))^2.\end{aligned} \tag{41}$$

Replacing f with χ_{I_1} and f' with χ_{I_2} in (28) shows that

$$\mathscr{E}(\Delta_1 Z_{ij}\cdot\Delta_2 Z_{ij})=0, \tag{42}$$

while replacing f and f' by χ_{I_k} shows

$$\begin{aligned}\mathscr{E}(\Delta_k Z_{ij})^2 &= \int_{I_k}\mathscr{E}\,Y_i\left(1-\frac{\Delta Q_{ij}}{1-H_{i-}}\right)\frac{dQ_{ij}}{1-H_{i-}}\\ &\leqq C\Delta_k Q_{ij}\qquad\text{because } (1-H_{i-})^{-1}\text{ is bounded on }[0,\tau]\\ &\leqq C(\Delta_k Q_{ij})^\alpha\qquad\text{for any }\alpha\in(0,1).\end{aligned} \tag{43}$$

Note that in the sequel the constant C may be different on each appearance; however, it can always be chosen not to depend on I_1 and I_2 though it will often depend on α. Substituting (42) and (43) back in (41), we see that to establish (40) it remains to suitably bound $\mathscr{E}(\Delta_1 Z_{ij}\cdot\Delta_2 Z_{ij})^2$. Now since $\Delta_k Z_{ij}=\Delta_k N_{ij}-\int_{I_k} Y_i(1-H_{i-})^{-1}dQ_{ij}$,

$$\begin{aligned}|\Delta_1 Z_{ij}\cdot\Delta_2 Z_{ij}| &\leqq \Delta_1 N_{ij}\cdot\Delta_2 N_{ij}+(\int_{I_1} Y_i(1-H_{i-})^{-1}dQ_{ij})(\int_{I_2} Y_i(1-H_{i-})^{-1}dQ_{ij})\\ &\quad+\Delta_1 N_{ij}(\int_{I_2} Y_i(1-H_{i-})^{-1}dQ_{ij})+\Delta_2 N_{ij}(\int_{I_1} Y_i(1-H_{i-})^{-1}dQ_{ij})\\ &\leqq C\,Y(0)^2,\end{aligned} \tag{44}$$

while expanding (42) we find that for $\alpha_1=1-\beta_1\in(0,1)$

$$\begin{aligned}&\mathscr{E}(\Delta_1 N_{ij}\cdot\Delta_2 N_{ij})+\mathscr{E}((\int_{I_1} Y_i(1-H_{i-})^{-1}dQ_{ij})\cdot(\int_{I_2} Y_i(1-H_{i-})^{-1}dQ_{ij}))\\ &\quad=\mathscr{E}(\Delta_1 N_{ij}\cdot(\int_{I_2} Y_i(1-H_{i-})^{-1}dQ_{ij}))+\mathscr{E}(\Delta_2 N_{ij}\cdot(\int_{I_1} Y_i(1-H_{i-})^{-1}dQ_{ij}))\\ &\quad\leqq C\cdot\mathscr{E}(\Delta_1 N_{ij}\cdot Y(0))\cdot\Delta_2 Q_{ij}+C\cdot\mathscr{E}(\Delta_2 N_{ij}\cdot Y(0))\cdot\Delta_1 Q_{ij}\\ &\quad\leqq C\cdot[\mathscr{E}((\Delta_1 N_{ij})^{\alpha_1}\cdot Y(0)^{1+\beta_1})\cdot(\Delta_2 Q_{ij})^{\alpha_1}\\ &\qquad+\mathscr{E}((\Delta_2 N_{ij})^{\alpha_1}\cdot Y(0)^{1+\beta_1})\cdot(\Delta_1 Q_{ij})^{\alpha_1}]\\ &\quad\leqq C[(\mathscr{E}(\Delta_1 N_{ij}))^{\alpha_1}(\mathscr{E}\,Y(0)^{1+1/\beta_1})^{\beta_1}(\Delta_2 Q_{ij})^{\alpha_1}\\ &\qquad+(\mathscr{E}(\Delta_2 N_{ij}))^{\alpha_1}(\mathscr{E}\,Y(0)^{1+1/\beta_1})^{\beta_1}(\Delta_1 Q_{ij})^{\alpha_1}]\\ &\quad\leqq C(\Delta_1 Q_{ij}\cdot\Delta_2 Q_{ij})^{\alpha_1}\end{aligned} \tag{45}$$

if

$$\mathscr{E}(Y(0))^{1+1/\beta_1} < \infty, \tag{46}$$

where we have used Hölder's inequality and the fact that

$$\mathscr{E}\,\Delta_k N_{ij} = \int_{I_k} \mathscr{E}\, Y_i (1 - H_{i-})^{-1}\, dQ_{ij}.$$

Therefore for $\alpha_2 = 1 - \beta_2 \in (0,1)$,

$$\begin{aligned}\mathscr{E}(\Delta_1 Z_{ij} \cdot \Delta_2 Z_{ij})^2 &= \mathscr{E}(|\Delta_1 Z_{ij} \cdot \Delta_2 Z_{ij}|^{\alpha_2} \cdot |\Delta_1 Z_{ij} \cdot \Delta_2 Z_{ij}|^{1+\beta_2}) \\ &\leqq C \cdot \mathscr{E}(|\Delta_1 Z_{ij} \cdot \Delta_2 Z_{ij}|^{\alpha_2} Y(0)^{2+2\beta_2}) \quad \text{(by (44))} \\ &\leqq C(\mathscr{E}\,|\Delta_1 Z_{ij} \cdot \Delta_2 Z_{ij}|)^{\alpha_2} (\mathscr{E}\, Y(0)^{2+2/\beta_2})^{\beta_2} \qquad (47)\end{aligned}$$

(by Hölder's inequality)

$$\leqq C(\Delta_1 Q_{ij} \cdot \Delta_2 Q_{ij})^{\alpha_1 \alpha_2}$$

by (44) and (45) if (46) holds and if

$$\mathscr{E}(Y(0)^{2+2/\beta_2}) < \infty. \tag{48}$$

Now by A 3* we can choose $\beta_1 < \frac{1}{6}$ such that (46) holds and $\beta_2 < \frac{2}{5}$ such that (48) holds. For such a choice $\alpha_1 \alpha_2 = (1-\beta_1)(1-\beta_2) > \frac{5}{6} \cdot \frac{3}{5} = \frac{1}{2}$ and therefore, by (47), for sufficiently small $\alpha > \frac{1}{2}$ we can find a $C < \infty$ such that

$$\mathscr{E}(\Delta_1 Z_{ij} \cdot \Delta_2 Z_{ij})^2 \leqq C(\Delta_1 Q_{ij} \cdot \Delta_2 Q_{ij})^{\alpha} \quad \text{for all } t_1, t, t_2, \tag{49}$$

which completes the proof. □

We have now shown that for τ_i satisfying $\mathscr{E}\, Y_i(\tau_i) > 0$,

$$\{n^{-\frac{1}{2}} Z_{ij}^n;\, i,j \leqq m\} \to_{\mathscr{D}} \{W_{ij};\, i,j \leqq m\} \quad \text{on } \prod_{i=1}^{m} D[0,\tau_i]^m$$

where the limit has been defined in the statement of Theorem 3, while jointly $\hat{H}_i^n - H_i \to_{\mathscr{D}} 0$ and $\hat{Q}_{ij}^n - Q_{ij} \to_{\mathscr{D}} 0$ each on $D[0,\tau_i]$. The next lemma on the Skorohod-Dudley construction is only stated and proved for a single process $n^{-\frac{1}{2}} Z_{ij}^n$, but the required simultaneous result can obviously be proved in the same way.

Lemma 4. *A Skorohod-Dudley construction is possible for $n^{-\frac{1}{2}} Z_{ij}^n$, $n = 1, 2, \ldots$ with respect to the supremum norm on $D[0,\tau_i]$ where τ_i satisfies $\mathscr{E}\, Y_i(\tau_i) > 0$.*

Proof. Write $W_n = n^{-\frac{1}{2}} Z_{ij}^n$, $n = 1, 2, \ldots$; $W = W_{ij}$; and $\tau = \tau_i$. Let $t_1, t_2, \ldots$ be an enumeration of the jump points of Q_{ij} in $[0,\tau]$ and let $\delta_k = \Delta Q_{ij}(t_k) \geqq 0$ for all k, $\sum_k \delta_k \leqq 1$. Define $v(t) = t + \sum_{t_k \leqq t} \delta_k$ and $\delta(t) = v(t) - v(t-) = \delta_k$ if $t = t_k$ for some $k = 1, 2, \ldots$ and $= 0$ otherwise. By the comments above, $W_n \to_{\mathscr{D}} W$ as $n \to \infty$ in $D[0,\tau]$ where W is a zero mean Gaussian process with independent increments and variance function $\operatorname{var}(W(t)) = \int_0^t f(s)\, dQ_{ij}(s) = A(t)$ say, for some bounded non-negative measurable function f on $[0,\tau]$. Let W^* be the zero mean

Gaussian process on $[0, v(\tau)]$ with independent increments and almost surely continuous sample paths, such that $\mathrm{var}(W^*(u)) = A(t-) + f(t)(u - v(t-))$ where t is the unique solution of $v(t-) \leqq u \leqq v(t)$. (W^* exists with these properties because $\mathrm{var}(W^*(u))$ is a continuous nondecreasing function of u.)

Now define, for $u \in [0, v(\tau)]$, $n = 1, 2, \ldots$ and $m = 1, 2, \ldots, \infty$,

$$W^*_{n,m}(u) = \begin{cases} W_n(t) & \text{if } u = v(t) \text{ for some } t \in [0, \tau] \\ W_n(t_k-) & \text{if } v(t_k-) \leqq u < v(t_k) \text{ for some } k > m \\ W_n(t_k-) + \dfrac{u - v(t_k-)}{\delta_k} \cdot (W_n(t_k) - W_n(t_k-)) & \\ & \text{if } v(t_k-) \leqq u < v(t_k) \text{ for some } k \leqq m \end{cases}$$

and define, also for $u \in [0, v(\tau)]$ and $m = 1, 2, \ldots, \infty$,

$$W^*_{\infty,m}(u) = \begin{cases} W^*(v(t)) & \text{if } u = v(t) \text{ for some } t \in [0, \tau] \\ W^*(v(t_k-) & \text{if } v(t_k-) \leqq u < v(t_k) \text{ for some } k > m \\ W^*(v(t_k-)) + \dfrac{u - v(t_k-)}{\delta_k} (W^*(v(t_k)) - W^*(v(t_k-)) & \\ & \text{if } v(t_k-) \leqq u < v(t_k) \text{ for some } k \leqq m. \end{cases}$$

In words, for $n = 1, 2, \ldots$, $W^*_{n,m}$ is obtained from W_n by inserting time intervals of length δ_k at t_k and joining $W_n(t_k-)$ to $W_n(t_k)$ by a straight line across this interval if $k \leqq m$, but continuing a horizontal line from $W_n(t_k-)$ if $k > m$; while $W^*_{\infty,m}$ is obtained from W^* in a similar way except that W^* has already been defined on the extended time interval. We see from this construction that $W^*_{\infty,m}$ is a random element of $D[0, v(\tau)]$ for $m < \infty$ while $W^*_{\infty,\infty}$ is a random element of $C[0, v(\tau)]$.

Now fixing $m < \infty$ for the moment, we can prove that $W^*_{n,m} \to_{\mathcal{D}} W^*_{\infty,m}$ as $n \to \infty$ in $D[0, v(\tau)]$. For the convergence of the finite dimensional distributions is again straightforward. Tightness is proved by proving tightness on each of the $2m+1$ intervals $[0, v(s_1-)]$, $[v(s_1-), v(s_1)]$, $\ldots$, $[v(s_m), v(\tau)]$ where $s_1, \ldots, s_m$ is $t_1, \ldots, t_m$ put into increasing order. Tightness on an interval of the form $[v(s_k-), v(s_k)]$ follows from convergence in distribution of $(W_n(s_k-), W_n(s_k))$ as $n \to \infty$, while tightness on $[v(s_{k-1}), v(s_k-)]$ follows from tightness of W_n (redefined in the points s_k as $W_n(s_k-)$) on $[s_{k-1}, s_k]$ and the observation that the modulus of continuity $w'_x(\delta)$ is smaller for $W^*_{n,m}$ on $[v(s_{k-1}), v(s_k-)]$ than it is for W_n on $[s_{k-1}, s_k]$.

Next we show that $\mathcal{E} \| W^*_{n,m} - W^*_{n,\infty} \|^2_{v(\tau)} \to 0$ as $m \to \infty$ uniformly in $n = 1, 2, \ldots, \infty$. For any n

$$\begin{aligned} \| W^*_{n,m} - W^*_{n,\infty} \|^2_{v(\tau)} &= \sup_{k > m} (W^*_{n,\infty}(v(t_k)) - W^*_{n,\infty}(v(t_k-)))^2 \\ &\leqq \sum_{k > m} (W^*_{n,\infty}(v(t_k)) - W^*_{n,\infty}(v(t_k-)))^2, \end{aligned}$$

which implies that

$$\mathcal{E} \| W^*_{n,m} - W^*_{n,\infty} \|^2_{v(\tau)} \leqq \sum_{k > m} f(t_k) \delta_k \to 0$$

as $m\to\infty$ uniformly in $n=1,2,\ldots,\infty$. Since the Skorohod d_0-distance on $D[0,v(\tau)]$ is smaller than the supremum norm distance, we have now shown that in $D[0,v(\tau)]$

$$W^*_{n,m}\to_{\mathscr{D}} W^*_{\infty,m} \quad \text{as } n\to\infty \text{ for } m=1,2,\ldots$$

and

$$W^*_{\infty,m}\to_{\mathscr{D}} W^*_{\infty,\infty} \quad \text{as } m\to\infty$$

$$\lim_{m\to\infty}\limsup_{n\to\infty} P(d_0(W^*_{n,m},W^*_{n,\infty})\geqq\varepsilon)=0 \quad \text{for all } \varepsilon>0.$$

So by Billingsley [3, Theorem 4.2]

$$W^*_{n,\infty}\to_{\mathscr{D}} W^*_{\infty,\infty} \quad \text{as } n\to\infty.$$

Since $W^*_{\infty,\infty}$ has almost surely continuous sample paths we can apply the Skorohod-Dudley theorem; and going back to the interval $[0,\tau]$ we have finally constructed W'_n, $n=1,2,\ldots$ and W' having the same marginal distributions as W_n *and* W but now defined on a single probability space and satisfying $\|W'_n-W'\|_\tau\to 0$ almost surely as $n\to\infty$. □

Lemma 5. *Let H_n, and Z_n, $n=1,2,\ldots$ and Z be random elements of $D[0,\tau]$ defined on a single probability space $(\Omega,\mathscr{F},P)$ and such that with probability* 1, *H_n and Z_n have paths of bounded variation and $Z_n(0)=0$ for each $n=1,2,\ldots$,* (∗) $\limsup_{n\to\infty}\int_{(0,\tau]}|dH_n(s)|<\infty$, $\|Z_n-Z\|\to 0$, *and* $\|H_n-h\|\to 0$ *where h is a fixed function of bounded variation on $[0,\tau]$. Then $\|\int H_{n-}\,dZ_n-\int h_-\,dZ\|\to 0$ and $\|\int H_n\,dZ_n-\int h\,dZ\|\to 0$ almost surely, where $\int h\,dZ$ is defined as $hZ-\int Z_-\,dh$ and $\int h_-\,dZ$ as $hZ-\int Z\,dh$ (because Z does not necessarily have paths of bounded variation). In fact to conclude $\|\int H_{n-}\,dZ_n-\int h_-\,dZ\|\to 0$ or $\|\int^- H_n\,dZ_n-\int^- h\,dZ\|\to 0$ almost surely we can weaken* (∗) *to* $\limsup_{n\to\infty}\int_{(0,\tau)}|dH_n(s)|<\infty$.

If furthermore stochastic integrals $h_-\circ Z$ and $h\circ Z$ (Meyer [11, definition 18]) *can be defined, then these coincide with $\int h_-\,dZ$ and $\int h\,dZ$, respectively.*

Proof. Fix an $\omega\in\Omega$ not in the exceptional event of probability zero specified above, and denote by H_n, Z_n and Z the functions on $[0,\tau]$: $H_n(\omega)$, $Z_n(\omega)$ and $Z(\omega)$. Choose an $\varepsilon>0$. Then there exists a $Z^*(=Z^*(\omega,\varepsilon))$ such that $Z^*(0)=0$, Z^* is of bounded variation, and $\|Z-Z^*\|<\varepsilon$. Next we write

$$\begin{aligned}\|\textstyle\int H_{n-}\,dZ_n-\int h_-\,dZ\| &\leqq \|\int H_{n-}\,dZ_n-\int H_{n-}\,dZ^*\|\\ &\quad+\|\int H_{n-}\,dZ^*-\int h_-\,dZ^*\|+\|\int h_-\,dZ^*-\int h_-\,dZ\|\\ &= A_n+B_n+C \quad \text{(say)}.\end{aligned}$$

Now using the formula $\int_{(0,t]}X(s-)\,dY(s)=\int_{(0,t)}(Y(t)-Y(s))\,dX(s)$ for functions X, Y in $D[0,\tau]$ of bounded variation and with $Y(0)=0$, and the fact that $\limsup_{n\to\infty}\int_{(0,\tau)}|dH_n(s)|<\infty$ it is easy to show that

$$\limsup_{n\to\infty} A_n=o(1) \quad \text{as } \varepsilon\downarrow 0$$

and that

$$C=o(1) \quad \text{as } \varepsilon\downarrow 0.$$

Finally $B_n \leqq \|H_n - h\| \int_{[0,\tau]} |dZ^*(s)| \to 0$ as $n \to \infty$, so combining these relationships we have the required result. Similar arguments establish the other assertions of almost sure convergence in norm.

Now we look at the second part of the lemma. By Meyer [11, IV n° 23], if $h_- \circ Z$ and $h \circ Z$ can be defined, $hZ = h_- \circ Z + Z \circ h$; by [11, IV n° 29], $Z \circ h = \int Z\, dh$. So $hZ - \int Z\, dh = h \circ Z$. Again by [11, IV n° 38] $hZ = h \circ Z + Z \circ h$ where according to [11, IV n° 29] $Z_- \circ h = \int Z_-\, dh$. So $hZ - \int Z_-\, dh = h \circ Z$. □

Corollary. *Suppose Lemma 5 allows us to conclude that almost surely*

$$\|\int H_n^{(i)}\, dZ_n - \int h^{(i)}\, dZ\| \to 0, \quad i = 1, \ldots, r$$

and

$$\|H_{n-}^{(i)}\, dZ_n - \int h_-^{(i)}\, dZ\| \to 0, \quad i = r+1, \ldots, s.$$

Then $\left\| \int \prod_{i=1}^{r} H_n^{(i)} \cdot \prod_{i=r+1}^{s} H_{n-}^{(i)}\, dZ_n - \int \prod_{i=1}^{r} h^{(i)} \prod_{i=r+1}^{s} h_-^{(i)}\, dZ \right\| \to 0$ *almost surely.*

Proof. Apply Lemma 5 first to $H_n^{(1)}$ and Z_n, then to $H_n^{(2)}$ and $\int H_n^{(1)}\, dZ_n$, etc. □

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Until now, Arne Jensen's writings have been scattered throughout Danish and international publications. To provide an overview of his contributions to traffic theory, operational research, futurology, planning economics, risk theory, and all related areas, a collection and translation for the first time into English was initiated by his friends and colleagues. This resulting work has been edited by the author's longstanding colleague Erik Johnsen, who has also written the Foreword.

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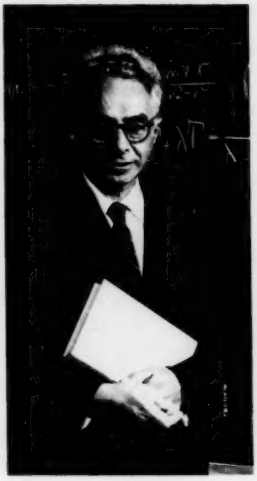

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Examples

Bombieri, E., Giusti, E.: Harnack's inequality for elliptic differential equations on minimal surfaces. Inventiones math. **15**, 24–46 (1971).

Tate, J.T.: *p*-Divisible Groups. In: Proceedings of a Conference on Local Fields, pp. 158–183. Berlin-Heidelberg-New York: Springer 1967.

B. Marking

1. Text

The words **"Theorem", "Lemma", "Corollary", "Proposition"** etc. are normally printed in **boldface,** followed by the formulation in italics (to be underlined in the manuscript).
The words "*Proof*", "*Remark*", "*Definition*", "*Note*" etc. are printed in *italics* with the formulation in ordinary typeface.
Words or sentences to be set in italics should be marked by single underlining.

2. Formulae

Letters in formulae are normally printed in italics, figures in ordinary typeface.
It will help the printer if in doubtful cases the position of indices and exponents is marked thus: $b_{\downarrow}$, $a^{\uparrow}$. Spacing of indices and exponents must be specially indicated ($A_m{}^n{}_n{}^m$) otherwise they will be set (A_{mn}^{nm}).
Underlining for special alphabets and typefaces should be done according to the following code:

single underlining:	small letter
double underlining:	capital letter
brown:	boldface headings, boldface letters in formulae
yellow:	upright abbreviations e. g. Re, Im, log, sin, ord, id, lim, sup, etc.)
red:	Greek
blue:	Gothic
green:	Script
violet:	the numeral 1, and zero (to distinguish them from the small letter *l* and the capital letter *O*)

The following are frequently confused:

$\cup$, $\mathfrak{u}$, $\bigcup$, U; $\circ$, o, O, 0; $\times$, x, X, κ; $\vee$, v, ν; $\theta, \Theta, \phi, \varphi, \Phi, \varnothing, \emptyset$; ψ, Ψ; $\varepsilon, \in$;

a', a^1; the symbol *a* and the indefinite article a;

also the handwritten Roman letters:

c, C; e, l; I, J; k, K; o, O; p, P; s, S; u, U; v, V; w, W; x, X; z, Z;

Please take care to distinguish them in some way.

C. Examples

1. Special alphabets or typefaces

Script	$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}, \mathscr{K}, \mathscr{L}, \mathscr{M}, \mathscr{N}, \mathscr{O}, \mathscr{P}, \mathscr{Q}, \mathscr{R}, \mathscr{S}, \mathscr{T}, \mathscr{U}, \mathscr{V}, \mathscr{W}, \mathscr{X}, \mathscr{Y}, \mathscr{Z}$ $\mathscr{a}, \mathscr{b}, \mathscr{c}, \mathscr{d}, \mathscr{e}, \mathscr{f}, \mathscr{g}, \mathscr{h}, \mathscr{i}, \mathscr{j}, \mathscr{k}, \mathscr{l}, \mathscr{m}, \mathscr{n}, \mathscr{o}, \mathscr{p}, \mathscr{q}, \mathscr{r}, \mathscr{s}, \mathscr{t}, \mathscr{u}, \mathscr{v}, \mathscr{w}, \mathscr{x}, \mathscr{y}, \mathscr{z}$
Sanserif	A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z
Gothic	$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{I}, \mathfrak{J}, \mathfrak{K}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}, \mathfrak{O}, \mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, \mathfrak{U}, \mathfrak{V}, \mathfrak{W}, \mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}, \mathfrak{m}, \mathfrak{n}, \mathfrak{o}, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z}$
Boldface	**A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z** **a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z**
Special Roman	$\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{H}, \mathbb{I}, \mathbb{J}, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{N}, \mathbb{O}, \mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{1}$
Greek	$\Gamma, \Delta, \Theta, \Lambda, \Xi, \Pi, \Sigma, \Phi, \Psi, \Omega$ $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \vartheta, \iota, \kappa, \lambda, \mu, \nu, \xi, o, \pi, \rho, \sigma, \tau, \upsilon, \varphi, \phi, \chi, \psi, \omega$

2. Notations

preferred form	instead of	preferred form	instead of
A^*, $\tilde{b}, \gamma', \mathfrak{v}, \mathbf{v}$	$\bar{A}, \hat{b}, \check{\gamma}, \vec{v}$	$f\colon A \to B$	$A \xrightarrow{f} B$
lim sup, lim inf	$\overline{\lim}, \underline{\lim}$		
inj lim, proj lim	$\varinjlim, \varprojlim$		
$\exp(-(x^2+y^2)/a^2)$	$e^{-\frac{x^2+y^2}{a^2}}$	$\dfrac{\cos(1/x)}{(a+b/x)^{1/2}}$	$\dfrac{\cos\frac{1}{x}}{\sqrt{a+\frac{b}{x}}}$
f^{-1}	$\overset{-1}{f}$		

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